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SCIENTIFIC PAPERS

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PREFACE

THIS volume completes the collection of my Father's published papers. The two last papers (Nos. 445 and 446) were left ready for the press, but were not sent to any channel of publication until after the Author's death.

Mr W. F. Sedgwick, late Scholar of Trinity College, Cambridge, who had done valuable service in sending corrections of my Father's writings during his lifetime, kindly consented to examine the proofs of the later papers of this volume [No. 399 onwards] which had not been printed off at the time of the Author's death. He has done this very thoroughly, checking the numerical calculations other than those embodied in tables, and supplying footnotes to elucidate doubtful or obscure points in the text. These notes are enclosed in square brackets [ ] and signed W. F. S. It has not been thought necessary to notice minor corrections.

RAYLEIGH.

Sept. 1920.

CONTENTS

* [1914. It would have been in better accordance with usage to have said "of Relative Index differing little from Unity."]

* [1917. It would be more correct to say $P_n(\cos\theta)$, where $\cos\theta$ lies between ± 1.]

ERRATA

(INCLUDING THE ERRATA NOTED IN VOLUME V. PAGE XIII.)

VOLUME I.

Page viii, line 4. } *For* end lies *read* ends lie.
,, 64, line 8. }

,, 86, last line. *For* 1882 *read* 1881.

,, 89, line 10. *Insert* comma *after* maximum.

,, 144, line 6 from bottom. *For* D *read* $D_{,}$.

,, 324, equation (8). *Insert* negative sign *before the single integral*. }

,, ,, line 2 from bottom. *For* (1) *read* (5). }

,, 325, line 10. *For* $-nVH$ *read* $+n\rho VH$. }

} And *Theory of Sound*, Vol. I. (1894), p. 477, equation (8) and last line, and p. 478, line 12.

,, 442, line 9. *After* $\frac{\rho' - \rho}{\rho'}$ *insert* y.

,, 443, line 9. *For* (7) *read* (8).

,, 443, line 10. *For* η *read* ξ.

,, 446, line 10. *For* ϕ *read* ϕ'.

,, 448, line 5. *For* v *read* c.

,, 459, line 17. *For* 256, 257 *read* 456, 457.

,, 492, line 7 from bottom. *For* $r\sqrt{2n}$ *read* $r/\sqrt{2n}$.

,, 494, lines 10 and 12. *For* $-\frac{2mr^2}{n^2-4m^2}\cos 2\theta$ *read* $+\frac{2mr^2}{n^2-4m^2}\cos 2\theta$.

,, 523, line 9. *For* n/λ *read* n/k.

,, 524. In the second term of equations (32) and following *for* ΔK^{-1} *read* $\Delta\mu^{-1}$.

,, 525, line 11. *For* f *read* f_1.

,, 526, line 13. *For* $f : g$ *read* $f_1 : g_1$.

,, 528, line 3 from bottom. *For* e^{int} *read* $e^{i(nt-kr_0)}$.

,, 538, line 11 from bottom. This passage is incorrect (*see* Vol. VI. Art. 355, p. 41).

,, 556. In line 8 *after* (15) *add* with $s\phi - \frac{1}{2}\pi$ for $s\phi$; in line 9 *for* δA_s *read* $\delta A_s'$; and *for* line 10 *substitute* $+\delta A_s'$ *as* $\{\cos\frac{1}{2}s\pi + \cos(\frac{1}{2}s\pi + s\pi)\} F$.
Throughout lines 12—25 *for* $A_s, A_1, A_2, \ldots A_6, \delta A_s$ *read* $A_s', A_1', A_2', \ldots A_6', \delta A_s'$; *for* $\sin\frac{1}{2}s\pi$ *read* $-\cos\frac{1}{2}s\pi$; and *reverse* the signs of the expressions for A_2', A_4', A_6'. Similarly, in *Theory of Sound*, Vol. I. (1894), p. 427, *substitute* $s\phi + \frac{1}{2}\pi$ *for* $s\phi$ in (32) (*see* p. 424), and in lines 11—26 *for* $A_s', A_s, \delta A_s$ *read* $A_s, A_s', \delta A_s'$, and *for* sin *read* +cos. Also in (43) and (47) *for* $s^2 - s$ *read* $s^3 - s$. [In both cases the work done corresponding to δA_s vanishes whether s be odd or even.]

VOLUME II.

,, 197, line 19. *For* nature *read* value.

,, 240, line 22. *For* dp/dx *read* dp/dy.

,, 241, line 2. *For* du/dx *read* du/dy.

,, 244, line 4. *For* k/n *read* n/k.

,, 323, lines 7 and 16 from bottom. *For* Thomson *read* C. Thompson.

,, 345, line 8 from bottom. *For* as pressures *read* at pressures.

,, 386, lines 12, 15, and 19. *For* cos CBD *read* cos CBB'.

,, 389, line 6. *For* minor *read* mirror.

,, 414, line 5. *For* favourable *read* favourably.

,, 551, first footnote. *For* 1866 *read* 1886.

VOLUME III.

Page 11, footnote. *For* has *read* have.
,, 92, line 4. *For* Vol. I. *read* Vol. II.
,, 129, equation (12). *For* $e^{u(i-x)}dx$ *read* $e^{u(i-x)}du$.
,, 162, line 19, and p. 224, second footnote. *For* Jellet *read* Jellett.
,, 179, line 15. *For* Provostaye *read* De la Provostaye.
,, 224, equation (20). *For* 2χ *read* χ.
,, ,, second footnote. *For* p. 179 *read* p. 343. } And *Theory of Sound*, Vol. I. (1894), p. 412, equation (12), and p. 423 (footnote).
,, 231, line 5 of first footnote. *For* 171 *read* 172.
,, 273, lines 15 and 20. *For* $\{\phi(x)\}^2$ *read* $\int_{-\infty}^{+\infty}\{\phi(x)\}^2 dx$.
,, 314, line 1. *For* (38) *read* (39).
,, 326. In the lower part of the Table, under Ampton *for* $c\flat+4$ *read* $e\flat+4$, and under Terling (3) *for* $b\flat+6$ *read* $b+6$ (and in *Theory of Sound*, Vol. I. (1894), p. 393).
,, 522, equation (31). *Insert as factor of last term* $1/R$.
,, 548, second footnote. *For* 1863 *read* 1868.
,, 569, second footnote. *For* alcohol *read* water.
,, 580, line 3. Prof. Orr remarks that a is a function of r.

VOLUME IV.

,, 14, lines 6 and 8. *For* 38 *read* 42.
,, 267, lines 6, 10, and 20, and p. 269, line 1. *For* van t' Hoff *read* van 't Hoff. Also in Index, p. 604 (the entry should be under Hoff).
,, 277, equation (12). *For* dz *read* dx.
,, 299, first footnote. *For* 1887 *read* 1877.
,, 369, footnote. *For* 1890 *read* 1896.
,, 400, equation (14). A formula equivalent to this was given by Lorenz in 1890.
,, 418. In table opposite 6 *for* ·354 *read* ·324.
,, 453, line 8 from bottom. *For* $\frac{2}{n-1}$ *read* $-\frac{2}{n-1}$.
,, 556, line 8 from bottom. *For* reflected *read* rotated.
,, 570, line 7 (Section III). *For* 176 *read* 179.
,, 576, line 7 from bottom. }
,, 586, line 20. } *For* end lies *read* ends lie.
,, 582, last line. *For* 557 *read* 555.
,, 603. *Transfer* the entry under Provostaye *to* De la Provostaye.
,, 604. *Transfer the entry* II 553 *from* W. Weber *to* H. F. Weber.

VOLUME V.

,, 43, line 19. *For* (5) *read* (2).
,, 137, line 14. μ is here used in two senses, which must be distinguished.
,, 149, line 3. *For* P_0 *read* P_1.
,, 209, footnote. *For* XLX. *read* XIX.
,, 241, line 10 from bottom. *For* position *read* supposition.
,, 255, first footnote. *For* Matthews *read* Mathews.
,, 256, line 6. *For* 1889 *read* 1899.
,, 265, line 16 from bottom. *For* § 351 *read* § 251.
,, ,, ,, 15 ,, ,, *For* solution *read* relation.
,, 266, lines 5 and 6, and *Theory of Sound*, § 251. An equivalent result had at an earlier date been obtained by De Morgan (*see* Volume VI. p. 233).
,, 286, line 7. *For* a *read* x.

VOLUME V—*continued.*

Page 364, title, and p. ix, Art. 320. *After* Acoustical Notes *add* VII.

,, 409, first line of P.S. *For* anwer *read* answer.

,, 444, line 2 of footnote. *For* p. 441, line 9 *read* p. 442, line 9.

,, 496, equation (4). *Substitute* equation (19) on p. 253 of Volume VI. (*see* pp. 251—253), *reading* $\left(\frac{l'}{l}-\frac{l}{l'}\right)$ *for* $\left(\frac{l'}{l}+\frac{l}{l'}\right)$.

,, 549, equation (48). *For* e^{-ikr} *read* e^{-ikr_0}.

,, 619, line 3. *Omit* the second expression for $J_n(n)$.
,, ,, lines 11, 12, 19. *For* 2·1123 *read* 1·3447.
,, ,, line 12. *For* 1·1814 *read* 1·8558.
,, ,, line 19. *For* ·51342 *read* ·8065.
See the first footnote on p. 211 of Volume VI.

VOLUME VI.

,, 4, first footnote. *After* equation (8) *add*:—*Scientific Papers*, Vol. V. p. 619. *See also* Errata last noted above.

,, 5, line 3. *For* $(2n+1)\,z^2=4n\,(n+1)\,(n+2)$ *read* $z^2=2n\,(n+2)$, so that z^2 is an integer.

,, 11, last footnote. *For* § 230 *read* § 250 (fourth edition).

,, 13, equation (17). *For* $\frac{4}{5}k^4a^4$ *read* $\frac{5}{4}k^4a^4$.

,, 14, footnote. *For* § 247 *read* § 251 (fourth edition).

,, 78, footnote. *Add*:—*Scientific Papers*, Vol. V. p. 400.

,, 87, footnote. *Add*:—Thomson and Tait's *Natural Philosophy*, Vol. I. p. 497.

,, 89, second footnote. *For* 328 *read* 329.

,, 90, second footnote. *Add*:—*Math. and Phys. Papers*, Vol. IV. p. 77.

,, 138, footnote. *For* 1868 *read* 1865, *and for* Vol. II. p. 128, *read* Vol. I. p. 526.

,, 148, footnote. *Add*:—*Scientific Papers*, Vol. IV. p. 407, and this Volume, p. 47.

,, 155, footnote. *For* Vol. IV. *read* Vol. III.

,, 222, second footnote. *For* Vol. II. *read* Vol. I. And in *Theory of Sound*, Vol. I. (1894), last line of § 207, *for* 4·4747 *read* 4·4774.

,, 223, line 5 from bottom. *For* 0·5772156 *read* 0·5772157.

,, 225, line 1. *For* much greater *read* not much greater.

,, ,, line 6 from bottom. *For* 13·094 *read* 3·3274.

,, 253, equation (19). *For* $\left(\frac{l'}{l}+\frac{l}{l'}\right)$ *read* $\left(\frac{l'}{l}-\frac{l}{l'}\right)$.

,, 259, line 5. *For* $-\frac{2}{a}\frac{dy}{dz}$ *read* $\mp\frac{2}{a}\frac{dy}{dz}$.

,, 263, equation (24). *For* $\frac{\omega^2 a}{2T}$ *read* $\frac{\omega^2 a^3}{2T}$.

,, ,, ,, (25). *For* $\left(1-\frac{3r^2}{a^2}\right)$ *read* $\left(1+\frac{3}{4}\frac{r^2}{a^2}\right)$.

,, 282, footnote. *For* p. 77 *read* p. 71.

,, 303, line 17. *For* $\sqrt{(bvc/\kappa)}$ *read* $\sqrt{(bvc\kappa)}$.

,, 307, line 8. *For* $\frac{d\phi}{dy}$ *read* $-\frac{d\phi}{dy}$.

,, 315, line 2. *Delete* 195.

,, 341, second footnote. *Add*:—[This Volume, p. 275].

,, 351, line 13 from bottom. *For* $Tg\rho$ *read* $T/g\rho$.

350.

NOTE ON BESSEL'S FUNCTIONS AS APPLIED TO THE VIBRATIONS OF A CIRCULAR MEMBRANE.

[*Philosophical Magazine*, Vol. XXI. pp. 53–58, 1911.]

IT often happens that physical considerations point to analytical conclusions not yet formulated. The pure mathematician will admit that arguments of this kind are suggestive, while the physicist may regard them as conclusive.

The first question here to be touched upon relates to the dependence of the roots of the function $J_n(z)$ upon the order n, regarded as susceptible of continuous variation. It will be shown that each root increases continually with n.

Let us contemplate the transverse vibrations of a membrane fixed along the radii $\theta = 0$ and $\theta = \beta$ and also along the circular arc $r = 1$. A typical simple vibration is expressed by*

$$w = J_n(z_n^{(s)} r) \, . \sin n\theta \, . \cos(z_n^{(s)} t), \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $z_n^{(s)}$ is a finite root of $J_n(z) = 0$, and $n = \pi/\beta$. Of these finite roots the lowest $z_n^{(1)}$ gives the principal vibration, *i.e.* the one without internal circular nodes. For the vibration corresponding to $z_n^{(s)}$ the number of internal nodal circles is $s - 1$.

As prescribed, the vibration (1) has no internal nodal diameter. It might be generalized by taking $n = \nu\pi/\beta$, where ν is an integer; but for our purpose nothing would be gained, since β is at disposal, and a suitable reduction of β comes to the same as the introduction of ν.

In tracing the effect of a diminishing β it may suffice to commence at $\beta = \pi$, or $n = 1$. The frequencies of vibration are then proportional to the roots of the function J_1. The reduction of β is supposed to be effected by

* *Theory of Sound*, §§ 205, 207.

increasing without limit the potential energy of the displacement (w) at every point of the small sector to be cut off. We may imagine suitable springs to be introduced whose stiffness is gradually increased, and that without limit. During this process every frequency originally finite must increase*, finally by an amount proportional to $d\beta$; and, as we know, no zero root can become finite. Thus before and after the change the finite roots correspond each to each, and every member of the latter series exceeds the corresponding member of the former.

As β continues to diminish this process goes on until when β reaches $\frac{1}{2}\pi$, n again becomes integral and equal to 2. We infer that every finite root of J_2 exceeds the corresponding finite root of J_1. In like manner every finite root of J_3 exceeds the corresponding root of J_2, and so on†.

I was led to consider this question by a remark of Gray and Mathews‡—"It seems probable that between every pair of successive real roots of J_n there is exactly one real root of J_{n+1}. It does not appear that this has been strictly proved; there must in any case be an odd number of roots in the interval." The property just established seems to allow the proof to be completed.

As regards the latter part of the statement, it may be considered to be a consequence of the well-known relation

$$J_{n+1}(z) = \frac{n}{z} J_n(z) - J_n'(z). \quad \text{..................(2)}$$

When J_n vanishes, J_{n+1} has the opposite sign to J_n', both these quantities being finite§. But at consecutive roots of J_n, J_n' must assume opposite signs, and so therefore must J_{n+1}. Accordingly the number of roots of J_{n+1} in the interval must be *odd*.

The theorem required then follows readily. For the first root of J_{n+1} must lie between the first and second roots of J_n. We have proved that it exceeds the first root. If it also exceeded the second root, the interval would be destitute of roots, contrary to what we have just seen. In like manner the second root of J_{n+1} lies between the second and third roots of J_n, and so on. The roots of J_{n+1} *separate* those of J_n ||.

* *Loc. cit.* §§ 88, 92*a*.

† [1915. Similar arguments may be applied to tesseral spherical harmonics, proportional to $\cos s\phi$, where ϕ denotes longitude, of fixed order n and continuously variable s.]

‡ *Bessel's Functions*, 1895, p. 50.

§ If J_n, J_{n+1} could vanish together, the sequence formula, (8) below, would require that every succeeding order vanish also. This of course is impossible, if only because when n is great the lowest root of J_n is of order of magnitude n.

|| I have since found in Whittaker's *Modern Analysis*, § 152, another proof of this proposition, attributed to Gegenbauer (1897).

The physical argument may easily be extended to show in like manner that all the finite roots of $J_n'(z)$ increase continually with n. For this purpose it is only necessary to alter the boundary condition at $r=1$ so as to make $dw/dr=0$ instead of $w=0$. The only difference in (1) is that $z_n^{(s)}$ now denotes a root of $J_n'(z)=0$. Mechanically the membrane is fixed as before along $\theta=0$, $\theta=\beta$, but all points on the circular boundary are free to slide transversely. The required conclusion follows by the same argument as was applied to J_n.

It is also true that there must be at least one root of J'_{n+1} between any two consecutive roots of J_n', but this is not so easily proved as for the original functions. If we differentiate (2) with respect to z and then eliminate J_n between the equation so obtained and the general differential equation, viz.

$$J_n'' + \frac{1}{z} J_n' + \left(1 - \frac{n^2}{z^2}\right) J_n = 0, \quad \text{......................(3)}$$

we find

$$\left(1 - \frac{n^2}{z^2}\right) J'_{n+1} + \frac{n}{z^3}(n^2 - 1 - z^2) J_n' + \left(1 - \frac{n^2+n}{z^2}\right) J_n'' = 0. \quad \text{...(4)}$$

In (4) we suppose that z is a root of J_n', so that $J_n'=0$. The argument then proceeds as before if we can assume that z^2-n^2 and $z^2-n(n+1)$ are both positive. Passing over this question for the moment, we notice that J_n'' and J'_{n+1} have opposite signs, and that both functions are finite. In fact if J_n'' and J_n' could vanish together, so also by (3) would J_n, and again by (2) J_{n+1}; and this we have already seen to be impossible.

At consecutive roots of J_n', J_n'' must have opposite signs, and therefore also J'_{n+1}. Accordingly there must be at least one root of J'_{n+1} between consecutive roots of J_n'. It follows as before that the roots of J'_{n+1} separate those of J_n'.

It remains to prove that z^2 necessarily exceeds $n(n+1)$. That z^2 exceeds n^2 is well known*, but this does not suffice. We can obtain what we require from a formula given in *Theory of Sound*, 2nd ed. § 339. If the finite roots taken in order be $z_1, z_2, \ldots z_s \ldots$, we may write

$$\log J_n'(z) = \text{const.} + (n-1)\log z + \Sigma \log(1 - z^2/z_s^2),$$

the summation including all finite values of z_s; or on differentiation with respect to z

$$\frac{J_n''(z)}{J_n'(z)} = \frac{n-1}{z} - \Sigma \frac{2z}{z_s^2 - z^2}.$$

This holds for all values of z. If we put $z=n$, we get

$$\Sigma \frac{2n}{z_s^2 - n^2} = 1, \quad \text{..............................(5)}$$

* Riemann's *Partielle Differentialgleichungen*; *Theory of Sound*, § 210.

since by (3)

$$J_n''(n) \div J_n'(n) = -n^{-1}.$$

In (5) all the denominators are positive. We deduce

$$\frac{z_1^2 - n^2}{2n} = 1 + \frac{z_1^2 - n^2}{z_2^2 - n^2} + \frac{z_1^2 - n^2}{z_3^2 - n^2} + \ldots > 1 \,; \quad \ldots\ldots\ldots\ldots\ldots(6)$$

and therefore

$$z_1^2 > n^2 + 2n > n(n+1).$$

Our theorems are therefore proved.

If a closer approximation to z_1^2 is desired, it may be obtained by substituting on the right of (6) $2n$ for $z_1^2 - n^2$ in the numerators and neglecting n^2 in the denominators. Thus

$$\frac{z_1^2 - n^2}{2n} > 1 + 2n\,(z_2^{-2} + z_3^{-2} + \ldots)$$

$$> 1 + 2n\left\{z_1^{-2} + z_2^{-2} + z_3^{-3} + \ldots - \frac{1}{n(n+2)}\right\}.$$

Now, as is easily proved from the ascending series for J_n',

$$z_1^{-2} + z_2^{-2} + z_3^{-2} + \ldots = \frac{n+2}{4n(n+1)};$$

so that finally

$$z_1^2 > n^2 + 2n + \frac{n^3}{(n+1)(n+2)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

When n is very great, it will follow from (7) that $z_1^2 > n^2 + 3n$. However the approximation is not close, for the ultimate form is*

$$z_1^2 = n^2 + [1{\cdot}6130]\, n^{4/3}.$$

As has been mentioned, the sequence formula

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

prohibits the simultaneous evanescence of J_{n-1} and J_n, or of J_{n-1} and J_{n+1}. The question arises—can Bessel's functions whose orders (supposed integral) differ by more than 2 vanish simultaneously? If we change n into $n+1$ in (8) and then eliminate J_n, we get

$$\left\{\frac{4n(n+1)}{z^2} - 1\right\} J_{n+1} = J_{n-1} + \frac{2n}{z} J_{n+2}, \quad \ldots\ldots\ldots\ldots(9)$$

from which it appears that if J_{n-1} and J_{n+2} vanish simultaneously, then either $J_{n+1} = 0$, which is impossible, or $z^2 = 4n(n+1)$. Any common root of J_{n-1} and J_{n+2} must therefore be such that its square is an integer.

* *Phil. Mag.* Vol. xx. p. 1003, 1910, equation (8). [1913. A correction is here introduced. See Nicholson, *Phil. Mag.* Vol. xxv. p. 200, 1913.]

Pursuing the process, we find that if J_{n-1}, J_{n+3} have a common root z, then

$$(2n+1)\,z^2 = 4n\,(n+1)\,(n+2),$$

so that z^2 is rational. And however far we go, we find that the simultaneous evanescence of two Bessel's functions requires that the common root be such that z^2 satisfies an algebraic equation whose coefficients are integers, the degree of the equation rising with the difference in order of the functions. If, as seems probable, a root of a Bessel's function cannot satisfy an integral algebraic equation, it would follow that no two Bessel's functions have a common root. The question seems worthy of the attention of mathematicians.

351.

HYDRODYNAMICAL NOTES.

[*Philosophical Magazine*, Vol. XXI. pp. 177—195, 1911.]

Potential and Kinetic Energies of Wave Motion.—Waves moving into Shallower Water.—Concentrated Initial Disturbance with inclusion of Capillarity.—Periodic Waves in Deep Water advancing without change of Type.—Tide Races.—Rotational Fluid Motion in a Corner.—Steady Motion in a Corner of Viscous Fluid.

IN the problems here considered the fluid is regarded as incompressible, and the motion is supposed to take place in two dimensions.

Potential and Kinetic Energies of Wave Motion.

When there is no dispersion, the energy of a progressive wave of any form is half potential and half kinetic. Thus in the case of a long wave in shallow water, "if we suppose that initially the surface is displaced, but that the particles have no velocity, we shall evidently obtain (as in the case of sound) two equal waves travelling in opposite directions, whose total energies are equal, and together make up the potential energy of the original displacement. Now the elevation of the derived waves must be half of that of the original displacement, and accordingly the potential energies less in the ratio of 4 : 1. Since therefore the potential energy of each derived wave is one quarter, and the total energy one half that of the original displacement, it follows that in the derived wave the potential and kinetic energies are equal" *.

The assumption that the displacement in each derived wave, when separated, is similar to the original displacement fails when the medium is dispersive. The equality of the two kinds of energy in an infinite progressive train of simple waves may, however, be established as follows.

* "On Waves," *Phil. Mag.* Vol. I. p. 257 (1876); *Scientific Papers*, Vol. I. p. 254.

Consider first an infinite series of simple stationary waves, of which the energy is at one moment wholly potential and [a quarter of] a period later wholly kinetic. If t denote the time and E the total energy, we may write

$$\text{K.E.} = E \sin^2 nt, \qquad \text{P.E.} = E \cos^2 nt.$$

Upon this superpose a similar system, displaced through a quarter wave-length in space and through a quarter period in time. For this, taken by itself, we should have

$$\text{K.E} = E \cos^2 nt, \qquad \text{P.E.} = E \sin^2 nt.$$

And, the vibrations being *conjugate*, the potential and kinetic energies of the combined motion may be found by simple addition of the components, and are accordingly independent of the time, and each equal to E. Now the resultant motion is a simple progressive train, of which the potential and kinetic energies are thus seen to be equal.

A similar argument is applicable to prove the equality of energies in the motion of a simple conical pendulum.

It is to be observed that the conclusion is in general limited to vibrations which are infinitely small.

Waves moving into Shallower Water.

The problem proposed is the passage of an infinite train of simple infinitesimal waves from deep water into water which shallows gradually in such a manner that there is no loss of energy by reflexion or otherwise. At any stage the whole energy, being the double of the potential energy, is proportional per unit length to the square of the height; and for motion in two dimensions the only remaining question for our purpose is what are to be regarded as corresponding lengths along the direction of propagation.

In the case of long waves, where the wave-length (λ) is long in comparison with the depth (l) of the water, corresponding parts are as the velocities of propagation (V), or since the periodic time (τ) is constant, as λ. Conservation of energy then requires that

$$(\text{height})^2 \times V = \text{constant}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or since V varies as $l^{\frac{1}{2}}$, height varies as $l^{-\frac{1}{4}}$*.

But for a dispersive medium corresponding parts are not proportional to V, and the argument requires modification. A uniform regime being established, what we are to equate at two separated places where the waves are of different character is the *rate of propagation of energy* through these places. It is a general proposition that in any kind of waves the ratio of the energy propagated past a fixed point in unit time to that resident in unit

* *Loc. cit.* p. 255.

length is U, where U is the *group-velocity*, equal to $d\sigma/dk$, where $\sigma = 2\pi/\tau$, $k = 2\pi/\lambda$*. Hence in our problem we must take

$$\text{height varies as } U^{-\frac{1}{2}}, \qquad (2)$$

which includes the former result, since in a non-dispersive medium $U = V$.

For waves in water of depth l,

$$\sigma^2 = gk \tanh kl, \qquad (3)$$

whence

$$2\sigma U/g = \tanh kl + kl\,(1 - \tanh^2 kl). \qquad (4)$$

As the wave progresses, σ remains constant, (3) determines k in terms of l, and U follows from (4). If we write

$$\sigma^2 l/g = l', \qquad (5)$$

(3) becomes

$$kl \,.\, \tanh kl = l', \qquad (6)$$

and (4) may be written

$$2\sigma U/g = kl + (l' - l'^2)/kl. \qquad (7)$$

By (6), (7) U is determined as a function of l' or by (5) of l.

If kl, and therefore l', is very great, $kl = l'$, and then by (7) if U_0 be the corresponding value of U,

$$2\sigma U_0/g = 1, \qquad (8)$$

and in general

$$U/U_0 = kl + (l' - l'^2)/kl. \qquad (9)$$

Equations (2), (5), (6), (9) may be regarded as giving the solution of the problem in terms of a known σ. It is perhaps more practical to replace σ in (5) by λ_0, the corresponding wave-length in a great depth. The relation between σ and λ_0 being $\sigma^2 = 2\pi g/\lambda_0$, we find in place of (5)

$$l' = 2\pi l/\lambda_0 = k_0 l. \qquad (10)$$

Starting in (10) from λ_0 and l we may obtain l', whence (6) gives kl, and (9) gives U/U_0. But in calculating results by means of tables of the hyperbolic functions it is more convenient to start from kl. We find

kl	l'	U/U_0	kl	l'	U/U_0
∞	kl	1·000	·6	·322	·964
10	kl	1·000	·5	·231	·855
5	4·999	1·001	·4	·152	·722
2	1·928	1·105	·3	·087	·566
1·5	1·358	1·176	·2	·039	·390
1·0	·762	1·182	·1	·010	·200
·8	·531	1·110	kl	$(kl)^2$	$2kl$
·7	·423	1·048	—	—	—

* *Proc. Lond. Math. Soc.* Vol. IX. 1877; *Scientific Papers*, Vol. I. p. 326.

It appears that U/U_0 does not differ much from unity between $l' = \cdot 23$ and $l' = \infty$, so that the shallowing of the water does not at first produce much effect upon the height of the waves. It must be remembered, however, that the wave-length is diminishing, so that waves, even though they do no more than maintain their height, grow *steeper*.

Concentrated Initial Disturbance with inclusion of Capillarity.

A simple approximate treatment of the general problem of initial linear disturbance is due to Kelvin*. We have for the elevation η at any point x and at any time t

$$\eta = \frac{1}{\pi}\int_0^\infty \cos kx \cos \sigma t \, dk$$

$$= \frac{1}{2\pi}\int_0^\infty \cos (kx - \sigma t)\, dk + \frac{1}{2\pi}\int_0^\infty \cos (kx + \sigma t)\, dk, \quad \text{......(1)}$$

in which σ is a function of k, determined by the character of the dispersive medium—expressing that the initial elevation ($t = 0$) is concentrated at the origin of x. When t is great, the angles whose cosines are to be integrated will in general vary rapidly with k, and the corresponding parts of the integral contribute little to the total result. The most important part of the range of integration is the neighbourhood of places where $kx \pm \sigma t$ is stationary with respect to k, *i.e.* where

$$x \pm t\frac{d\sigma}{dk} = 0. \quad \text{..............................(2)}$$

In the vast majority of practical applications $d\sigma/dk$ is positive, so that if x and t are also positive the second integral in (1) makes no sensible contribution. The result then depends upon the first integral, and only upon such parts of that as lie in the neighbourhood of the value, or values, of k which satisfy (2) taken with the lower sign. If k_1 be such a value, Kelvin shows that the corresponding term in η has an expression equivalent to

$$\eta = \frac{\cos(\sigma_1 t - k_1 x - \frac{1}{4}\pi)}{\sqrt{\{-2\pi t\, d^2\sigma/dk_1^2\}}}, \quad \text{......................(3)}$$

σ_1 being the value of σ corresponding to k_1.

In the case of deep-water waves where $\sigma = \sqrt{(gk)}$, there is only one predominant value of k for given values of x and t, and (2) gives

$$k_1 = gt^2/4x^2, \qquad \sigma_1 = gt/2x, \quad \text{....................(4)}$$

making

$$\sigma_1 t - k_1 x - \tfrac{1}{4}\pi = gt^2/4x - \tfrac{1}{4}\pi, \quad \text{......................(5)}$$

and finally

$$\eta = \frac{g^{\frac{1}{2}} t}{2\pi^{\frac{1}{2}} x^{\frac{3}{2}}} \cos\left\{\frac{gt^2}{4x} - \frac{\pi}{4}\right\}, \quad \text{......................(6)}$$

the well-known formula of Cauchy and Poisson.

* *Proc. Roy. Soc.* Vol. XLII. p. 80 (1887); *Math. and Phys. Papers*, Vol. IV. p. 303.

In the numerator of (3) σ_1 and k_1 are functions of x and t. If we inquire what change (Λ) in x with t constant alters the angle by 2π, we find

$$\Lambda \left\{k_1 + \left(x - t\frac{d\sigma}{dk_1}\right)\frac{dk_1}{dx}\right\} = 2\pi,$$

so that by (2) $\Lambda = 2\pi/k_1$, *i.e.* the effective wave-length Λ coincides with that of the predominant component in the original integral (1), and a like result holds for the periodic time*. Again, it follows from (2) that $k_1x - \sigma_1 t$ in (3) may be replaced by $\int k_1 dx$, as is exemplified in (4) and (6).

When the waves move under the influence of a capillary tension T in addition to gravity,

$$\sigma^2 = gk + Tk^3/\rho, \qquad \text{.............................(7)}$$

ρ being the density, and for the wave-velocity (V)

$$V^2 = \sigma^2/k^2 = g/k + Tk/\rho, \qquad \text{..........................(8)}$$

as first found by Kelvin. Under these circumstances V has a minimum value when

$$k^2 = g\rho/T. \qquad \text{................................(9)}$$

The group-velocity U is equal to $d\sigma/dk$, or to $d\,(kV)/dk$; so that when V has a minimum value, U and V coincide. Referring to this, Kelvin towards the close of his paper remarks "The working out of our present problem for this case, or any case in which there are either minimums or maximums, or both maximums and minimums, of wave-velocity, is particularly interesting, but time does not permit of its being included in the present communication."

A glance at the simplified form (3) shows, however, that the special case arises, not when V is a minimum (or maximum), but when U is so, since then $d^2\sigma/dk_1^2$ vanishes. As given by (3), η would become infinite—an indication that the approximation must be pursued. If $k = k_1 + \xi$, we have in general in the neighbourhood of k_1,

$$kx - \sigma t = k_1 x - \sigma_1 t + \left(x - t\frac{d\sigma}{dk_1}\right)\xi - \frac{t}{1\,.\,2}\frac{d^2\sigma}{dk_1^2}\xi^2 - \frac{t}{1\,.\,2\,.\,3}\frac{d^3\sigma}{dk_1^3}\xi^3. \quad \text{...(10)}$$

In the present case where the term in ξ^2 disappears, as well as that in ξ, we get in place of (3) when t is great

$$\eta = \frac{\cos(k_1x - \sigma_1 t)}{2\pi\,\{\frac{1}{6}t\,d^3\sigma/dk_1^3\}^{\frac{1}{3}}}\int_{-\infty}^{+\infty}\cos\alpha^3\,.\,d\alpha, \qquad \text{..............(11)}$$

varying as $t^{-\frac{1}{3}}$ instead of as $t^{-\frac{1}{2}}$.

The definite integral is included in the general form

$$\int_{-\infty}^{+\infty}\cos\alpha^m\,.\,d\alpha = \frac{2}{m}\,\Gamma\left(\frac{1}{m}\right)\cos\frac{\pi}{2m}, \qquad \text{..................(12)}$$

* Cf. Green, *Proc. Roy. Soc. Ed.* Vol. xxix. p. 445 (1909).

giving

$$\int_{-\infty}^{+\infty} \cos \alpha^2 \,.\, d\alpha = \sqrt{\left(\frac{\pi}{2}\right)}\,; \qquad \int_{-\infty}^{+\infty} \cos \alpha^3 \,.\, d\alpha = \frac{1}{\sqrt{3}}\,\Gamma\left(\tfrac{1}{3}\right). \quad \ldots\ldots(13)$$

The former is employed in the derivation of (3).

The occurrence of stationary values of U is determined from (7) by means of a quadratic. There is but one such value (U_0), easily seen to be a minimum, and it occurs when

$$k^2 = \{\sqrt{\tfrac{4}{3}} - 1\}\frac{g\rho}{T} = \cdot 1547\,\frac{g\rho}{T}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

On the other hand, the minimum of V occurs when $k^2 = g\rho/T$ simply.

When t is great, there is no important effect so long as x (positive) is less than $U_0 t$. For this value of x the Kelvin formula requires the modification expressed by (11). When x is decidedly greater than $U_0 t$, there arise two terms of the Kelvin form, indicating that there are now two systems of waves of different wave-lengths, effective at the same place.

It will be seen that the introduction of capillarity greatly alters the character of the solution. The quiescent region inside the annular waves is easily recognized a few seconds after a very small stone is dropped into smooth water*, but I have not observed the duplicity of the annular waves themselves. Probably the capillary waves of short wave-length are rapidly damped, especially when the water-surface is not quite clean. It would be interesting to experiment upon truly linear waves, such as might be generated by the sudden electrical charge or discharge of a wire stretched just above the surface. But the full development of the peculiar features to be expected on the inside of the wave-system seems to require a space larger than is conveniently available in a laboratory.

Periodic Waves in Deep Water advancing without change of Type.

The solution of this problem when the height of the waves is infinitesimal has been familiar for more than a century, and the pursuance of the approximation to cover the case of moderate height is to be found in a well-known paper by Stokes†. In a supplement published in 1880‡ the same author treated the problem by another method in which the space coordinates x, y are regarded as functions of ϕ, ψ the velocity and stream functions, and carried the approximation a stage further.

In an early publication§ I showed that some of the results of Stokes' first memoir could be very simply derived from the expression for the

* A checkered background, *e.g.* the sky seen through foliage, shows the waves best.

† *Camb. Phil. Soc. Trans.* Vol. VIII. p. 441 (1847); *Math. and Phys. Papers*, Vol. I. p. 197.

‡ *Loc. cit.* Vol. I. p. 314.

§ *Phil. Mag.* Vol. I. p. 257 (1876); *Scientific Papers*, Vol. I. p. 262. See also Lamb's *Hydrodynamics*, § 230.

stream-function in terms of x and y, and lately I have found that this method may be extended to give, as readily if perhaps less elegantly, all the results of Stokes' Supplement.

Supposing for brevity that the wave-length is 2π and the velocity of propagation unity, we take as the expression for the stream-function of the waves, reduced to rest,

$$\psi = y - \alpha e^{-y}\cos x - \beta e^{-2y}\cos 2x - \gamma e^{-3y}\cos 3x, \quad \text{..........(1)}$$

in which x is measured horizontally and y vertically downwards. This expression evidently satisfies the differential equation to which ψ is subject, whatever may be the values of the constants α, β, γ. From (1) we find

$$\begin{aligned} U^2 - 2gy &= (d\psi/dx)^2 + (d\psi/dy)^2 - 2gy \\ &= 1 - 2\psi + 2(1-g)\,y + 2\beta e^{-2y}\cos 2x + 4\gamma e^{-3y}\cos 3x \\ &\quad + \alpha^2 e^{-2y} + 4\beta^2 e^{-4y} + 9\gamma^2 e^{-6y} + 4\alpha\beta e^{-3y}\cos x \\ &\quad + 6\alpha\gamma e^{-4y}\cos 2x + 12\beta\gamma e^{-5y}\cos x. \quad \text{..........(2)} \end{aligned}$$

The condition to be satisfied at a free surface is the constancy of (2).

The solution to a moderate degree of approximation (as already referred to) may be obtained with omission of β and γ in (1), (2). Thus from (1) we get, determining ψ so that the mean value of y is zero,

$$y = \alpha(1 + \tfrac{5}{8}\alpha^2)\cos x - \tfrac{1}{2}\alpha^2\cos 2x + \tfrac{3}{8}\alpha^3\cos 3x, \quad \text{..........(3)}$$

which is correct as far as α^3 inclusive.

If we call the coefficient of $\cos x$ in (3) a, we may write with the same approximation

$$y = a\cos x - \tfrac{1}{2}a^2\cos 2x + \tfrac{3}{8}a^3\cos 3x. \quad \text{..........(4)}$$

Again from (2) with omission of β, γ,

$$U^2 - 2gy = \text{const.} + 2(1 - g - \alpha^2 - \alpha^4)\,y + \alpha^4\cos 2x - \tfrac{4}{3}\alpha^5\cos 3x. \quad \text{......(5)}$$

It appears from (5) that the surface condition may be satisfied with α only, provided that α^4 is neglected and that

$$1 - g - \alpha^2 = 0. \quad \text{..........(6)}$$

In (6) α may be replaced by a, and the equation determines the velocity of propagation. To exhibit this we must restore generality by introduction of $k(=2\pi/\lambda)$ and c the velocity of propagation, hitherto treated as unity. Consideration of "dimensions" shows that (6) becomes

$$kc^2 - g - a^2c^2k^3 = 0, \quad \text{..........(7)}$$

or

$$c^2 = g/k\,.\,(1 + k^2a^2). \quad \text{..........(8)}$$

Formulæ (4) and (8) are those given by Stokes in his first memoir.

By means of β and γ the surface condition (2) can be satisfied with inclusion of α^4 and α^5, and from (5) we see that β is of the order α^4 and γ of

the order α^5. The terms to be retained in (2), in addition to those given in (5), are

$$2\beta(1-2y)\cos 2x + 4\gamma\cos 3x + 4\alpha\beta\cos x$$
$$= 2\beta\cos 2x - 2\alpha\beta(\cos x + \cos 3x) + 4\gamma\cos 3x + 4\alpha\beta\cos x.$$

Expressing the terms in $\cos x$ by means of y, we get finally

$$U^2 - 2gy = \text{const.} + 2y(1 - g - \alpha^2 - \alpha^4 + \beta)$$
$$+ (\alpha^4 + 2\beta)\cos 2x + (4\gamma - \tfrac{4}{3}\alpha^5 - 2\alpha\beta)\cos 3x. \quad \text{......(9)}$$

In order to satisfy the surface condition of constant pressure, we must take

$$\beta = -\tfrac{1}{2}\alpha^4, \qquad \gamma = \tfrac{1}{12}\alpha^5, \quad \text{........................(10)}$$

and in addition

$$1 - g - \alpha^2 - \tfrac{3}{2}\alpha^4 = 0, \quad \text{..........................(11)}$$

correct to α^5 inclusive. The expression (1) for ψ thus assumes the form

$$\psi = y - \alpha e^{-y}\cos x + \tfrac{1}{2}\alpha^4 e^{-2y}\cos 2x - \tfrac{1}{12}\alpha^5 e^{-3y}\cos 3x, \quad \text{.........(12)}$$

from which y may be calculated in terms of x as far as α^5 inclusive.

By successive approximation, determining ψ so as to make the mean value of y equal to zero, we find as far as α^4

$$y = (\alpha + \tfrac{5}{8}\alpha^3)\cos x - (\tfrac{1}{2}\alpha^2 + \tfrac{4}{3}\alpha^4)\cos 2x + \tfrac{3}{8}\alpha^3\cos 3x - \tfrac{1}{3}\alpha^4\cos 4x, \quad \text{...(13)}$$

or, if we write as before a for the coefficient of $\cos x$,

$$y = a\cos x - (\tfrac{1}{2}a^2 + \tfrac{17}{24}a^4)\cos 2x + \tfrac{3}{8}a^3\cos 3x - \tfrac{1}{3}a^4\cos 4x, \quad \text{...(14)}$$

in agreement with equation (20) of Stokes' Supplement.

Expressed in terms of a, (11) becomes

$$g = 1 - a^2 - \tfrac{1}{4}a^4, \quad \text{.............................(15)}$$

or on restoration of k, c,

$$g = kc^2 - k^3a^2c^2 - \tfrac{1}{4}k^5a^4c^2. \quad \text{.......................(16)}$$

Thus the extension of (8) is

$$c^2 = g/k\,.\,(1 + k^2a^2 + \tfrac{4}{5}k^4a^4), \quad \text{.......................(17)}$$

which also agrees with Stokes' Supplement.

If we pursue the approximation one stage further, we find from (12) terms in α^5, additional to those expressed in (13). These are

$$y = \alpha^5\left\{\frac{373}{6\,.\,32}\cos x + \frac{243}{128}\cos 3x + \frac{125}{12\,.\,32}\cos 5x\right\}. \quad \text{...... (18)*}$$

It is of interest to compare the potential and kinetic energies of waves

* [1916. Burnside (*Proc. Lond. Math. Soc.* Vol. xv. p. 26, 1916) throws doubts upon the utility of Stokes' series.]

that are not infinitely small. For the stream-function of the waves regarded as progressive, we have, as in (1),

$$\psi = -\alpha e^{-y}\cos(x - ct) + \text{terms in } \alpha^4,$$

so that

$$(d\psi/dx)^2 + (d\psi/dy)^2 = \alpha^2 e^{-2y} + \text{terms in } \alpha^5.$$

Thus the mean kinetic energy per length x measured in the direction of propagation is

$$\frac{\alpha^2}{2}\int dx \int_y^\infty e^{-2y}\,dy = \frac{\alpha^2}{4}\int dx\, e^{-2y} = \frac{\alpha^2}{4}\int dx\,(1 - 2y + 2y^2) = \frac{\alpha^2}{4}\left\{x + 2\int y^2 dx\right\},$$

where y is the ordinate of the surface. And by (3)

$$\int y^2 dx = \{\tfrac{1}{2}(\alpha^2 + \tfrac{5}{4}\alpha^4) + \tfrac{1}{8}\alpha^4\}\, x.$$

Hence correct to α^4,

$$\text{K.E.} = \tfrac{1}{4}\alpha^2(1 + \alpha^2)\, x. \qquad (19)$$

Again, for the potential energy

$$\text{P.E.} = \tfrac{1}{2}g\int y^2\,dx = \tfrac{1}{2}gx(\tfrac{1}{2}\alpha^2 + \tfrac{3}{4}\alpha^4);$$

or since $g = 1 - \alpha^2$,

$$\text{P.E.} = \tfrac{1}{4}\alpha^2(1 + \tfrac{1}{2}\alpha^2)\, x. \qquad (20)$$

The kinetic energy thus exceeds the potential energy, when α^4 is retained.

Tide Races.

It is, I believe, generally recognized that seas are apt to be exceptionally heavy when the tide runs against the wind. An obvious explanation may be founded upon the fact that the relative motion of air and water is then greater than if the latter were not running, but it seems doubtful whether this explanation is adequate.

It has occurred to me that the cause may be rather in the motion of the stream relatively to itself, *e.g.* in the more rapid movement of the upper strata. Stokes' theory of the highest possible wave shows that in non-rotating water the angle at the crest is 120° and the height only moderate. In such waves the surface strata have a mean motion forwards. On the other hand, in Gerstner and Rankine's waves the fluid particles retain a mean position, but here there is *rotation* of such a character that (in the absence of waves) the surface strata have a relative motion backwards, *i.e.* against the direction of propagation*. It seems possible that waves moving against the tide may approximate more or less to the Gerstner type and thus be capable of acquiring a greater height and a sharper angle than would otherwise be expected. Needless to say, it is the steepness of waves, rather than their

* Lamb's *Hydrodynamics*, § 247.

mere height, which is a source of inconvenience and even danger to small craft.

The above is nothing more than a suggestion. I do not know of any detailed account of the special character of these waves, on which perhaps a better opinion might be founded.

Rotational Fluid Motion in a Corner.

The motion of incompressible inviscid fluid is here supposed to take place in two dimensions and to be bounded by two fixed planes meeting at an angle α. If there is no rotation, the stream-function ψ, satisfying $\nabla^2\psi = 0$, may be expressed by a series of terms

$$r^{\pi/\alpha} \sin \pi\theta/\alpha, \qquad r^{2\pi/\alpha} \sin 2\pi\theta/\alpha, \ldots \; r^{n\pi/\alpha} \sin n\pi\theta/\alpha,$$

where n is an integer, making $\psi = 0$ when $\theta = 0$ or $\theta = \alpha$. In the immediate vicinity of the origin the first term predominates. For example, if the angle be a right angle,

$$\psi = r^2 \sin 2\theta = 2xy, \qquad \text{.........................(1)}$$

if we introduce rectangular coordinates.

The possibility of irrotational motion depends upon the fixed boundary not being closed. If $\alpha < \pi$, the motion near the origin is finite; but if $\alpha > \pi$, the velocities deduced from ψ become infinite.

If there be rotation, motion may take place even though the boundary be closed. For example, the circuit may be completed by the arc of the circle $r = 1$. In the case which it is proposed to consider the rotation ω is *uniform*, and the motion may be regarded as steady. The stream-function then satisfies the general equation

$$\nabla^2\psi = d^2\psi/dx^2 + d^2\psi/dy^2 = 2\omega, \qquad \text{....................(2)}$$

or in polar coordinates

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} = 2\omega. \qquad \text{....................(3)}$$

When the angle is a right angle, it might perhaps be expected that there should be a simple expression for ψ in powers of x and y, analogous to (1) and applicable to the immediate vicinity of the origin; but we may easily satisfy ourselves that no such expression exists*. In order to express the motion we must find solutions of (3) subject to the conditions that $\psi = 0$ when $\theta = 0$ and when $\theta = \alpha$.

For this purpose we assume, as we may do, that

$$\psi = \Sigma R_n \sin n\pi\theta/\alpha, \qquad \text{..........................(4)}$$

* In strictness the satisfaction of (2) *at* the origin is inconsistent with the evanescence of ψ on the rectangular axes.

where n is integral and R_n a function of r only; and in deducing $\nabla^2\psi$ we may perform the differentiations with respect to θ (as well as with respect to r) under the sign of summation, since $\psi = 0$ at the limits. Thus

$$\nabla^2\psi = \Sigma\left(\frac{d^2R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} - \frac{n^2\pi^2}{\alpha^2 r^2}R_n\right)\sin\frac{n\pi\theta}{\alpha}. \quad \text{...........(5)}$$

The right-hand member of (3) may also be expressed in a series of sines of the form

$$2\omega = 8\omega/\pi \,.\, \Sigma n^{-1}\sin n\pi\theta/\alpha, \quad \text{......................(6)}$$

where n is an *odd* integer; and thus for all values of n we have

$$r^2\frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr} - \frac{n^2\pi^2 R_n}{\alpha^2} = \frac{4\omega}{n\pi}\{1-(-1)^n\}. \quad \text{...........(7)}$$

The general solution of (7) is

$$R_n = A_n r^{n\pi/\alpha} + B_n r^{-n\pi/\alpha} + \frac{4\omega\alpha^2 r^2\{1-(-1)^n\}}{n\pi(4\alpha^2 - n^2\pi^2)}, \quad \text{...........(8)}$$

the introduction of which into (4) gives ψ.

In (8) A_n and B_n are arbitrary constants to be determined by the other conditions of the problem. For example, we might make R_n, and therefore ψ, vanish when $r = r_1$ and when $r = r_2$, so that the fixed boundary enclosing the fluid would consist of two radii vectores and two circular arcs. If the fluid extend to the origin, we must make $B_n = 0$; and if the boundary be completed by the circular arc $r = 1$, we have $A_n = 0$ when n is even, and when n is odd

$$A_n = \frac{8\omega\alpha^2}{n\pi(4\alpha^2 - n^2\pi^2)} = 0. \quad \text{......................(9)}$$

Thus for the fluid enclosed in a circular sector of angle α and radius unity

$$\psi = 8\omega\alpha^2\,\Sigma\,\frac{r^{n\pi/\alpha} - r^2}{n\pi(n^2\pi^2 - 4\alpha^2)}\sin\frac{n\pi\theta}{\alpha}, \quad \text{..............(10)}$$

the summation extending to all odd integral values of n.

The above formula (10) relates to the motion of uniformly *rotating* fluid bounded by *stationary* radii vectores at $\theta = 0$, $\theta = \alpha$. We may suppose the containing vessel to have been rotating for a long time and that the fluid (under the influence of a very small viscosity) has acquired this rotation so that the whole revolves like a solid body. The motion expressed by (10) is that which would ensue if the rotation of the vessel were suddenly stopped. A related problem was solved a long time since by Stokes*, who considered the *irrotational* motion of fluid in a *revolving* sector. The solution of Stokes' problem is derivable from (10) by mere addition to the latter of $\psi_0 = -\frac{1}{2}\omega r^2$, for then $\psi + \psi_0$ satisfies $\nabla^2(\psi + \psi_0) = 0$; and this is perhaps the simplest

* *Camb. Phil. Trans.* Vol. VIII. p. 533 (1847); *Math. and Phys. Papers*, Vol. I. p. 305.

method of obtaining it. The results are in harmony; but the fact is not immediately apparent, inasmuch as Stokes expresses the motion by means of the velocity-potential, whereas here we have employed the stream-function.

That the subtraction of $\frac{1}{2}\omega r^2$ makes (10) an harmonic function shows that the series multiplying r^2 can be summed. In fact

$$8\alpha^2 \Sigma \frac{\sin(n\pi\theta/\alpha)}{n\pi(n^2\pi^2 - 4\alpha^2)} = \frac{\cos(2\theta - \alpha)}{2\cos\alpha} - \frac{1}{2},$$

so that
$$\psi/\omega = \tfrac{1}{2}r^2 - \frac{r^2\cos(2\theta-\alpha)}{2\cos\alpha} + 8\alpha^2 \Sigma \frac{r^{n\pi/\alpha}\sin n\pi\theta/\alpha}{n\pi(n^2\pi^2 - 4\alpha^2)}. \quad \text{......(11)}$$

In considering the character of the motion defined by (11) in the immediate vicinity of the origin we see that if $\alpha < \frac{1}{2}\pi$, the term in r^2 preponderates even when $n = 1$. When $\alpha = \frac{1}{2}\pi$ exactly, the second term in (11) and the first term under Σ corresponding to $n = 1$ become infinite, and the expression demands transformation. We find in this case

$$\psi/\omega = \tfrac{1}{2}r^2 + \frac{2r^2}{\pi}(\theta - \tfrac{1}{4}\pi)\cos 2\theta + r^2\sin 2\theta\left(\frac{2}{\pi}\log r - \frac{3}{2\pi}\right) + \frac{2}{\pi}\Sigma\frac{r^{2n}\sin 2n\theta}{n(n^2-1)}, \quad \text{.........(12)}$$

the summation commencing at $n = 3$. On the middle line $\theta = \frac{1}{4}\pi$, we have

$$\psi/\omega = \tfrac{1}{2}r^2 - \frac{3r^2}{2\pi} + \frac{2}{\pi}\left\{r^2\log r - \frac{r^6}{3.8} + \frac{r^{10}}{5.24} - \dots\right\}. \quad \text{......(13)}$$

The following are derived from (13):

r	$-\frac{1}{2}\pi\psi$	r	$-\frac{1}{2}\pi\psi$	r	$-\frac{1}{2}\pi\psi$
0·0	·00000	0·4	·14112	0·8	·13030
0·1	·02267	0·5	·16507	0·9	·07641
0·2	·06296	0·6	·17306	1·0	·00000
0·3	·10521	0·7	·16210		

The maximum value occurs when $r = ·592$. At the point $r = ·592$, $\theta = \frac{1}{4}\pi$, the fluid is stationary.

A similar transformation is required when $\alpha = 3\pi/2$.

When $\alpha = \pi$, the boundary becomes a semicircle, and the leading term ($n = 1$) is

$$\psi/\omega = -\frac{8}{3\pi}r\sin\theta = -\frac{3}{8\pi}y\dots, \quad \text{.................(14)}$$

which of itself represents an irrotational motion.

When $\alpha = 2\pi$, the two bounding radii vectores coincide and the containing vessel becomes a circle with a single partition wall at $\theta = 0$. In this case again the leading term is irrotational, being

$$\psi/\omega = -\frac{32}{15\pi} r^{\frac{3}{2}} \sin \tfrac{1}{2}\theta. \quad \text{.........................(15)}$$

Steady Motion in a Corner of a Viscous Fluid.

Here again we suppose the fluid to be incompressible and to move in two dimensions free from external forces, or at any rate from such as cannot be derived from a potential. If in the same notation as before ψ represents the stream-function, the general equation to be satisfied by ψ is

$$\nabla^4 \psi = 0\,; \quad \text{.................................(1)}$$

with the conditions that when $\theta = 0$ and $\theta = \alpha$,

$$\psi = 0, \qquad d\psi/d\theta = 0. \quad \text{...........................(2)}$$

It is worthy of remark that the problem is analytically the same as that of a plane elastic plate clamped at $\theta = 0$ and $\theta = \alpha$, upon which (in the region considered) no external forces act.

The general problem thus represented is one of great difficulty, and all that will be attempted here is the consideration of one or two particular cases. We inquire what solutions are possible such that ψ, as a function of r (the radius vector), is proportional to r^m. Introducing this supposition into (1), we get

$$\left\{m^2 + \frac{d^2}{d\theta^2}\right\}\left\{(m-2)^2 + \frac{d^2}{d\theta^2}\right\} \psi = 0, \quad \text{.................(3)}$$

as the equation determining the dependence on θ. The most general value of ψ consistent with our suppositions is thus

$$\psi = r^m \{A \cos m\theta + B \sin m\theta + C \cos (m-2)\,\theta + D \sin (m-2)\,\theta\}, \quad \text{...(4)}$$

where A, B, C, D are constants.

Equation (4) may be adapted to our purpose by taking

$$m = n\pi/\alpha, \quad \text{.................................(5)}$$

where n is an integer. Conditions (2) then give

$$A + C = 0, \qquad A + C \cos 2\alpha - D \sin 2\alpha = 0,$$

$$\frac{n\pi}{\alpha} B + \left(\frac{n\pi}{\alpha} - 2\right) D = 0,$$

$$\frac{n\pi}{\alpha} B + \left(\frac{n\pi}{\alpha} - 2\right) C \sin 2\alpha + \left(\frac{n\pi}{\alpha} - 2\right) D \cos 2\alpha = 0.$$

When we substitute in the second and fourth of these equations the values of A and B, derived from the first and third, there results

$$C(1-\cos 2\alpha)+D\sin 2\alpha=0,$$
$$C\sin 2\alpha-D(1-\cos 2\alpha)=0;$$

and these can only be harmonized when $\cos 2\alpha=1$, or $\alpha=s\pi$, where s is an integer. In physical problems, α is thus limited to the values π and 2π. To these cases (4) is applicable with C and D arbitrary, provided that we make

$$A+C=0,\qquad B+\left(1-\frac{2s}{n}\right)D=0. \quad\text{...............(5 bis)}$$

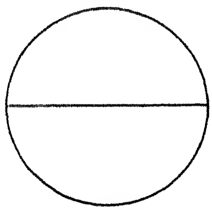
Thus
$$\psi=Cr^{n/s}\left\{\cos\left(\frac{n\theta}{s}-2\theta\right)-\cos\frac{n\theta}{s}\right\}$$
$$+Dr^{n/s}\left\{\sin\left(\frac{n\theta}{s}-2\theta\right)-\left(1-\frac{2s}{n}\right)\sin\frac{n\theta}{s}\right\}, \quad\text{.........(6)}$$

making
$$\nabla^2\psi=4\left(\frac{n}{s}-1\right)r^{-2+n/s}\left\{C\cos\left(\frac{n\theta}{s}-2\theta\right)+D\sin\left(\frac{n\theta}{s}-2\theta\right)\right\}. \quad\text{...(7)}$$

When $s=1$, $\alpha=\pi$, the corner disappears and we have simply a straight boundary (fig. 1). In this case $n=1$ gives a nugatory result. When $n=2$, we have

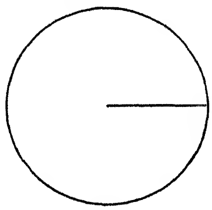
$$\psi=Cr^2(1-\cos 2\theta)=2Cy^2, \quad\text{.....................(8)}$$

Fig. 1.



$s=1$

Fig. 2.



$s=2$

and $\nabla^2\psi=4C$. When $n=3$,

$$\psi=Cr^3(\cos\theta-\cos 3\theta)+Dr^3(\sin\theta-\tfrac{1}{3}\sin 3\theta), \quad\text{.........(9)}$$
$$\nabla^2\psi=8r(C\cos\theta+D\sin\theta)=8(Cx+Dy). \quad\text{...............(10)}$$

In rectangular coordinates

$$\psi=4Cxy^2+\tfrac{4}{3}Dy^3, \quad\text{...........................(11)}$$

solutions which obviously satisfy the required conditions.

When $s=2$, $\alpha=2\pi$, the boundary consists of a straight wall extending from the origin in *one* direction (fig. 2). In this case (6) and (7) give

$$\psi=Cr^{\frac{1}{2}n}\{\cos(\tfrac{1}{2}n\theta-2\theta)-\cos\tfrac{1}{2}n\theta\}$$
$$+Dr^{\frac{1}{2}n}\left\{\sin(\tfrac{1}{2}n\theta-2\theta)-\left(1-\frac{4}{n}\right)\sin\tfrac{1}{2}n\theta\right\}, \quad\text{......(12)}$$

$$\nabla^2\psi=(2n-4)r^{\frac{1}{2}n-2}\{C\cos(\tfrac{1}{2}n\theta-2\theta)+D\sin(\tfrac{1}{2}n\theta-2\theta)\}. \quad\text{...(13)}$$

Solutions of interest are afforded in the case $n=1$. The C-solution is $(C=\frac{1}{4})$

$$\psi=\tfrac{1}{4}r^{\frac{1}{2}}(\cos\tfrac{3}{2}\theta-\cos\tfrac{1}{2}\theta)=-r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta, \quad \text{.........(14)}$$

vanishing when $\theta=\pi$, as well as when $\theta=0$, $\theta=2\pi$, and for no other admissible value of θ. The values of ψ are reversed when we write $2\pi-\theta$ for θ. As expressed, this value is negative from 0 to π and positive from π to 2π. The minimum occurs when $\theta=109°\,28'$. Every stream-line which enters the circle $(r=1)$ on the left of this radius leaves it on the right.

The velocities, represented by $d\psi/dr$ and $r^{-1}d\psi/d\theta$, are infinite at the origin.

For the D-solution we may take

$$\psi=r^{\frac{1}{2}}\sin^3\tfrac{1}{2}\theta. \quad \text{..............................(15)}$$

Here ψ retains its value unaltered when $2\pi-\theta$ is substituted for θ. When r is given, ψ increases continuously from $\theta=0$ to $\theta=\pi$. On the line $\theta=\pi$ the motion is entirely transverse to it. This is an interesting example of the flow of viscous fluid round a sharp corner. In the application to an elastic plate ψ represents the displacement at any point of the plate, supposed to be clamped along $\theta=0$, and otherwise free from force within the region considered. The following table exhibits corresponding values of r and θ such as to make $\psi=1$ in (15):

θ	r	θ	r
180°	1·00	60°	64·0
150°	1·23	20°	$10^4\times 3{\cdot}65$
120°	2·37	10°	$10^6\times 2{\cdot}28$
90°	8·00	0°	∞

When $n=2$, (12) appears to have no significance.

When $n=3$, the dependence on θ is the same as when $n=1$. Thus (14) and (15) may be generalized:

$$\psi=(Ar^{\frac{1}{2}}+Br^{\frac{3}{2}})\cos\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta, \quad \text{..................(16)}$$

$$\psi=(A'r^{\frac{1}{2}}+B'r^{\frac{3}{2}})\sin^3\tfrac{1}{2}\theta. \quad \text{..........................(17)}$$

For example, we could satisfy either of the conditions $\psi=0$, or $d\psi/dr=0$, on the circle $r=1$.

For $n=4$ the D-solution becomes nugatory; but for the C-solution we have

$$\psi=Cr^2(1-\cos 2\theta)=2Cr^2\sin^2\theta=2Cy^2. \quad \text{...........(18)}$$

The wall (or in the elastic plate problem the clamping) along $\theta=0$ is now without effect.

It will be seen that along these lines nothing can be done in the apparently simple problem of a horizontal plate clamped along the rectangular axes of x and y, if it be supposed free from force*. Ritz† has shown that the solution is not developable in powers of x and y, and it may be worth while to extend the proposition to the more general case when the axes, still regarded as lines of clamping, are inclined at any angle α. In terms of the now oblique coordinates x, y the general equation takes the form

$$(d^2/dx^2 + d^2/dy^2 - 2\cos\alpha\, d^2/dx\,dy)^2 w = 0, \quad \text{...........(19)}$$

which may be differentiated any number of times with respect to x and y, with the conditions

$$w = 0, \qquad dw/dy = 0, \qquad \text{when } y = 0, \quad \text{..............(20)}$$

$$w = 0, \qquad dw/dx = 0, \qquad \text{when } x = 0. \quad \text{..............(21)}$$

We may differentiate, as often as we please, (20) with respect to x and (21) with respect to y.

From these data it may be shown that at the origin *all* differential coefficients of w with respect to x and y vanish. The evanescence of those of zero and first order is expressed in (20), (21). As regards those of the second order we have from (20) $d^2w/dx^2 = 0$, $d^2w/dx\,dy = 0$, and from (21) $d^2w/dy^2 = 0$. Similarly for the third order from (20)

$$d^3w/dx^3 = 0, \qquad d^3w/dx^2dy = 0,$$

and from (21)

$$d^3w/dy^3 = 0, \qquad d^3w/dx\,dy^2 = 0.$$

For the fourth order (20) gives

$$d^4w/dx^4 = 0, \qquad d^4w/dx^3dy = 0,$$

and (21) gives

$$d^4w/dy^4 = 0, \qquad d^4w/dx\,dy^3 = 0.$$

So far d^4w/dx^2dy^2 might be finite, but (19) requires that it also vanish. This process may be continued. For the $m+1$ coefficients of the mth order we obtain four equations from (20), (21) and $m-3$ by differentiations of (19), so that all the differential coefficients of the mth order vanish. It follows that every differential coefficient of w with respect to x and y vanishes at the origin. I apprehend that the conclusion is valid for all angles α less than 2π. That the displacement at a distance r from the corner should diminish rapidly with r is easily intelligible, but that it should diminish more rapidly than any power of r, however high, would, I think, not have been expected without analytical proof.

* If indeed gravity act, $w = x^2y^2$ is a very simple solution.

† *Ann. d. Phys.* Bd. XXVIII. p. 760, 1909.

352.

ON A PHYSICAL INTERPRETATION OF SCHLÖMILCH'S THEOREM IN BESSEL'S FUNCTIONS.

[*Philosophical Magazine*, Vol. XXI. pp. 567—571, 1911.]

THIS theorem teaches that any function $f(r)$ which is finite and continuous for real values of r between the limits $r = 0$ and $r = \pi$, both inclusive, may be expanded in the form

$$f(r) = a_0 + a_1 J_0(r) + a_2 J_0(2r) + a_3 J_0(3r) + \ldots, \quad \ldots\ldots\ldots(1)$$

J_0 being the Bessel's function usually so denoted; and Schlömilch's demonstration has been reproduced with slight variations in several text-books*. So far as I have observed, it has been treated as a purely analytical development. From this point of view it presents rather an accidental appearance; and I have thought that a physical interpretation, which is not without interest in itself, may help to elucidate its origin and meaning.

The application that I have in mind is to the theory of aerial vibrations. Let us consider the most general vibrations in one dimension ξ which are periodic in time 2π and are also symmetrical with respect to the origins of ξ and t. The condensation s, for example, may be expressed

$$s = b_0 + b_1 \cos\xi \cos t + b_2 \cos 2\xi \cos 2t + \ldots, \quad \ldots\ldots\ldots\ldots(2)$$

where the coefficients b_0, b_1, &c. are arbitrary. (For simplicity it is supposed that the velocity of propagation is unity.) When $t = 0$, (2) becomes a function of ξ only, and we write

$$F(\xi) = b_0 + b_1 \cos\xi + b_2 \cos 2\xi + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

in which $F(\xi)$ may be considered to be an arbitrary function of ξ from 0 to π. Outside these limits F is determined by the equations

$$F(-\xi) = F(\xi + 2\pi) = F(\xi). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

* See, for example, Gray and Mathews' *Bessel's Functions*, p. 30; Whittaker's *Modern Analysis*, § 165.

We now superpose an infinite number of components, analogous to (2) with the same origins of space and time, and differing from one another only in the direction of ξ, these directions being limited to the plane xy, and in this plane distributed uniformly. The resultant is a function of t and r only, where $r = \sqrt{(x^2 + y^2)}$, independent of the third coordinate z, and therefore (as is known) takes the form

$$s = a_0 + a_1 J_0(r) \cos t + a_2 J_0(2r) \cos 2t + a_3 J_0(3r) \cos 3t + \ldots, \quad \ldots(5)$$

reducing to (1) when $t = 0$*. The expansion of a function in the series (1) is thus definitely suggested as probable in all cases and certainly possible in an immense variety. And it will be observed that no value of ξ greater than π contributes anything to the resultant, so long as $r < \pi$.

The relation here implied between F and f is of course identical with that used in the purely analytical investigation. If ϕ be the angle between ξ and any radius vector r to a point where the value of f is required, $\xi = r \cos \phi$, and the *mean* of all the components $F(\xi)$ is expressed by

$$f(r) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} F(r \cos \phi)\, d\phi. \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The solution of the problem of expressing F by means of f is obtained analytically with the aid of Abel's theorem. And here again a physical, or rather geometrical, interpretation throws light upon the process.

Equation (6) is the result of averaging $F(\xi)$ over all directions indifferently in the xy plane. Let us abandon this restriction and take the average when ξ is indifferently distributed in all directions whatever. The result now becomes a function only of R, the radius vector in space. If θ be the angle between R and one direction of ξ, $\xi = R \cos \theta$, and we obtain as the mean

$$\int_0^{\frac{1}{2}\pi} F(R \cos \theta) \sin \theta \, d\theta = \frac{1}{R} \{F_1(R) - F_1(0)\}, \quad \ldots\ldots\ldots\ldots(7)$$

where $F_1' = F$.

This result is obtained by a direct integration of $F(\xi)$ over all directions in space. It may also be arrived at indirectly from (6). In the latter $f(r)$ represents the averaging of $F(\xi)$ for all directions in a certain plane, the result being independent of the coordinate perpendicular to the plane. If we take the average again for all possible positions of this plane, we must recover (7). Now if θ be the angle between the normal to this plane and the radius vector R, $r = R \sin \theta$, and the mean is

$$\int_0^{\frac{1}{2}\pi} f(R \sin \theta) \sin \theta \, d\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

* It will appear later that the a's and b's are equal.

We conclude that

$$R\int_0^{\frac{1}{2}\pi} f(R\sin\theta)\sin\theta\, d\theta = F_1(R) - F_1(0), \qquad \text{...........(9)}$$

which may be considered as expressing F in terms of f.

If in (6), (9) we take $F(R) = \cos R$, we find*

$$\int_0^{\frac{1}{2}\pi} J_0(R\sin\theta)\sin\theta\, d\theta = R^{-1}\sin R.$$

Differentiating (9), we get

$$F(R) = \int_0^{\frac{1}{2}\pi} f(R\sin\theta)\sin\theta\, d\theta + R\int_0^{\frac{1}{2}\pi} f'(R\sin\theta)(1-\cos^2\theta)\, d\theta. \quad \text{...(10)}$$

Now

$$R\int_0^{\frac{1}{2}\pi} \cos^2\theta\, f'(R\sin\theta)\, d\theta = \int \cos\theta \,.\, df(R\sin\theta)$$

$$= -f(0) + \int_0^{\frac{1}{2}\pi} f(R\sin\theta)\sin\theta\, d\theta.$$

Accordingly $$F(R) = f(0) + R\int_0^{\frac{1}{2}\pi} f'(R\sin\theta)\, d\theta. \qquad \text{...............(11)}$$

That $f(r)$ in (1) may be arbitrary from 0 to π is now evident. By (3) and (6)

$$f(r) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} d\phi\, \{b_0 + b_1\cos(r\cos\phi) + b_2\cos(2r\cos\phi) + \ldots\}$$

$$= b_0 + b_1 J_0(r) + b_2 J_0(2r) + \ldots, \qquad \text{..............................(12)}$$

where $$b_0 = \frac{1}{\pi}\int_0^{\pi} F(\xi)\, d\xi, \quad b_n = \frac{2}{\pi}\int_0^{\pi} \cos n\xi\, F(\xi)\, d\xi. \qquad \text{.........(13)}$$

Further, with use of (11)

$$b_0 = f(0) + \frac{1}{\pi}\int_0^{\pi} d\xi \,.\, \xi \,.\, \int_0^{\frac{1}{2}\pi} f'(\xi\sin\theta)\, d\theta, \qquad \text{...........(14)}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} d\xi \,.\, \xi\cos n\xi \,.\, \int_0^{\frac{1}{2}\pi} f'(\xi\sin\theta)\, d\theta, \qquad \text{...........(15)}$$

by which the coefficients in (12) are completely expressed when f is given between 0 and π.

The physical interpretation of Schlömilch's theorem in respect of two-dimensional aerial vibrations is as follows:—Within the cylinder $r = \pi$ it is possible by suitable movements at the boundary to maintain a symmetrical motion which shall be strictly periodic in period 2π, and which at times $t = 0$, $t = 2\pi$, &c. (when there is no velocity), shall give a condensation which

* *Enc. Brit.* Art. "Wave Theory," 1888; *Scientific Papers*, Vol. III. p. 98.

is arbitrary over the whole of the radius. And this motion will maintain itself without external aid if outside $r=\pi$ the initial condition is chosen in accordance with (6), $F(\xi)$ for values of ξ greater than π being determined by (4). A similar statement applies of course to the vibrations of a stretched membrane, the transverse displacement w replacing s in (5).

Reference may be made to a simple example quoted by Whittaker. Initially let $f(r)=r$, so that from 0 to π the form of the membrane is conical. Then from (12), (14), (15)

$$b_0=\frac{\pi^2}{4},\quad b_n=0\ (n \text{ even}),\quad b_n=-\frac{2}{n^2}\ (n \text{ odd});$$

and thus

$$f(r)=\frac{\pi^2}{4}-2\left\{J_0(r)+\frac{1}{9}J_0(3r)+\frac{1}{25}J_0(5r)+\dots\right\}, \quad \dots\dots(16)$$

the right-hand member being equal to r from $r=0$ to $r=\pi$.

The corresponding vibration is of course expressed by (16) if we multiply each function $J_0(nr)$ by the time-factor $\cos nt$.

If this periodic vibration is to be maintained without external force, the initial condition must be such that it is represented by (16) for all values of r, and not merely for those less than π. By (11) from 0 to π, $F(\xi)=\frac{1}{2}\pi\xi$, from which again by (4) the value of F for higher values of ξ follows. Thus from π to 2π, $F(\xi)=\frac{1}{2}\pi(2\pi-\xi)$; from 2π to 3π, $F(\xi)=\frac{1}{2}\pi(\xi-2\pi)$; and so on. From these f is to be found by means of (6). For example, from π to 2π,

$$f(r)=r\int_0^{\sin\theta=\pi/r}\sin\theta\,d\theta+\int_{\sin\theta=\pi/r}^{\sin\theta=1}(2\pi-r\sin\theta)\,d\theta$$

$$=r-2\sqrt{(r^2-\pi^2)}+2\pi\cos^{-1}(\pi/r), \quad \dots\dots\dots\dots\dots\dots\dots(17)$$

where $\cos^{-1}(\pi/r)$ is to be taken in the first quadrant.

It is hardly necessary to add that a theorem similar to that proved above holds for aerial vibrations which are symmetrical in all directions about a centre. Thus within the sphere of radius π it is possible to have a motion which shall be strictly periodic and is such that the condensation is initially arbitrary at all points along the radius.

353.

BREATH FIGURES.

[*Nature*, Vol. LXXXVI. pp. 416, 417, 1911.]

THE manner in which aqueous vapour condenses upon ordinarily clean surfaces of glass or metal is familiar to all. Examination with a magnifier shows that the condensed water is in the form of small lenses, often in pretty close juxtaposition. The number and thickness of these lenses depend upon the cleanness of the glass and the amount of water deposited. In the days of wet collodion every photographer judged of the success of the cleaning process by the uniformity of the dew deposited from the breath.

Information as to the character of the deposit is obtained by looking through it at a candle or small gas flame. The diameter of the halo measures the angle at which the drops meet the glass, an angle which diminishes as the dew evaporates. That the flame is seen at all in good definition is a proof that some of the glass is uncovered. Even when both sides of a plate are dewed the flame is still seen distinctly though with much diminished intensity.

The process of formation may be followed to some extent under the microscope, the breath being led through a tube. The first deposit occurs very suddenly. As the condensation progresses, the drops grow, and many of the smaller ones coalesce. During evaporation there are two sorts of behaviour. Sometimes the boundaries of the drops contract, leaving the glass bare. In other cases the boundary of a drop remains fixed, while the thickness of the lens diminishes until all that remains is a thin lamina. Several successive formations of dew will often take place in what seems to be precisely the same pattern, showing that the local conditions which determine the situation of the drops have a certain degree of permanence.

An interesting and easy experiment has been described by Aitken (*Proc. Ed. Soc.* p. 94, 1893). Clean a glass plate in the usual way until the breath deposits equally.

"If we now pass over this clean surface the point of a blow-pipe flame, using a very small jet, and passing it over the glass with sufficient quickness to prevent the sudden heating breaking it; and if we now breathe on the glass after it is cold, we shall find the track of the flame clearly marked. While most of the surface looks white by the light reflected from the deposited moisture, the track of the flame is quite black; not a ray of light is scattered by it. It looks as if there were no moisture condensed on that part of the plate, as it seems unchanged; but if it be closely examined by a lens, it will be seen to be quite wet. But the water is so evenly distributed, that it forms a thin film, in which, with proper lighting and the aid of a lens, a display of interference colours may be seen as the film dries and thins away."

"Another way of studying the change produced on the surface of the glass by the action of the flame is to take the [plate], as above described, after a line has been drawn over it with the blow-pipe jet, and when cold let a drop of water fall on any part of it where it showed white when breathed on. Now tilt the plate to make the drop flow, and note the resistance to its flow, and how it draws itself up in the rear, leaving the plate dry. When, however, the moving drop comes to the part acted on by the flame, all resistance to flow ceases, and the drop rapidly spreads itself over the whole track, and shows a decided disinclination to leave it."

The impression thus produced lasts for some days or weeks, with diminishing distinctness. A permanent record may be obtained by the deposit of a very thin coat of silver by the usual chemical method. The silver attaches itself by preference to the track of the flame, and especially to the *edges* of the track, where presumably the combustion is most intense. It may be protected with celluloid, or other, varnish.

The view, expressed by Mr Aitken, which would attribute the effect to very fine dust deposited on the glass from the flame, does not commend itself to me. And yet mere heat is not very effective. I was unable to obtain a good result by strongly heating the *back* of a thin glass in a Bunsen flame. For this purpose a long flame on Ramsay's plan is suitable, especially if it be long enough to include the entire width of the plate.

It seems to me that we must appeal to varying degrees of cleanliness for the explanation, cleanliness meaning mainly freedom from grease. And one of the first things is to disabuse our minds of the idea that anything wiped with an ordinary cloth can possibly be clean. This subject was ably treated many years ago by Quincke (*Wied. Ann.* II. p. 145, 1877), who, however, seems to have remained in doubt whether a film of air might not give rise to the same effects as a film of grease. Quincke investigated the maximum edge-angle possible when a drop of liquid stands upon the surface of a solid. In general, the cleaner the surface, the smaller the

maximum edge-angle. With alcohol and petroleum there was no difficulty in reducing the maximum angle to zero. With water on glass the angle could be made small, but increased as time elapsed after cleaning.

As a detergent Quincke employed hot sulphuric acid. A few drops may be poured upon a thin glass plate, which is then strongly heated over a Bunsen burner. When somewhat cooled, the plate may be washed under the tap, rinsed with distilled water, and dried over the Bunsen without any kind of wiping. The parts wetted by the acid then behave much as the track of the blow-pipe flame in Aitken's experiment.

An even better treatment is with hydrofluoric acid, which actually renews the surface of the glass. A few drops of the commercial acid, diluted, say, ten times, may be employed, much as the sulphuric acid, only without heat. The parts so treated condense the breath in large laminæ, contrasting strongly with the ordinary deposit.

It must be admitted that some difficulties remain in attributing the behaviour of an ordinary plate to a superficial film of grease. One of these is the comparative permanence of breath figures, which often survive wiping with a cloth. The thought has sometimes occurred to me that the film of grease is not entirely superficial, but penetrates in some degree into the substance of the glass. In that case its removal and renewal would not be so easy. We know but little of the properties of matter in thin films, which may differ entirely from these of the same substance in mass. It may be recalled that a film of oil, one or two millionths of a millimetre thick, suffices to stop the movements of camphor on the surface of water, and that much smaller quantities may be rendered evident by optical and other methods.

354.

ON THE MOTION OF SOLID BODIES THROUGH VISCOUS LIQUID.

[*Philosophical Magazine,* Vol. XXI. pp. 697—711, 1911.]

§ 1. THE problem of the uniform and infinitely slow motion of a sphere, or cylinder, through an unlimited mass of incompressible viscous liquid otherwise at rest was fully treated by Stokes in his celebrated memoir on Pendulums*. The two cases mentioned stand in sharp contrast. In the first a relative steady motion of the fluid is easily determined, satisfying all the conditions both at the surface of the sphere and at infinity; and the force required to propel the sphere is found to be finite, being given by the formula (126)

$$-F = 6\pi\mu a V, \quad \text{.............................(1)}$$

where μ is the viscosity, a the radius, and V the velocity of the sphere. On the other hand in the case of the cylinder, moving transversely, no such steady motion is possible. If we suppose the cylinder originally at rest to be started and afterwards maintained in uniform motion, finite effects are propagated to ever greater and greater distances, and the motion of the fluid approaches no limit. Stokes shows that more and more of the fluid tends to accompany the travelling cylinder, which thus experiences a continually decreasing resistance.

§ 2. In attempting to go further, one of the first questions to suggest itself is whether similar conclusions are applicable to bodies of other forms. The consideration of this subject is often facilitated by use of the well-known analogy between the motion of a viscous fluid, when the square of the motion is neglected, and the displacements of an elastic solid. Suppose that in the latter case the solid is bounded by two closed surfaces, one of which completely envelopes the other. Whatever displacements (α, β, γ) be imposed at these two surfaces, there must be a corresponding configuration

* *Camb. Phil. Trans.* Vol. IX. 1850; *Math. and Phys. Papers,* Vol. III. p. 1

of equilibrium, satisfying certain differential equations. If the solid be incompressible, the otherwise arbitrary boundary displacements must be chosen subject to this condition. The same conclusion applies in two dimensions, where the bounding surfaces reduce to cylinders with parallel generating lines. For our present purpose we may suppose that at the outer surface the displacements are zero.

The contrast between the three-dimensional and two-dimensional cases arises when the outer surface is made to pass off to infinity. In the former case, where the inner surface is supposed to be limited in all directions, the displacements there imposed diminish, on receding from it, in such a manner that when the outer surface is removed to a sufficient distance no further sensible change occurs. In the two-dimensional case the inner surface extends to infinity, and the displacement affects sensibly points however distant, provided the outer surface be still further and sufficiently removed.

The nature of the distinction may be illustrated by a simple example relating to the conduction of heat through a uniform medium. If the temperature v be unity on the surface of the sphere $r=a$, and vanish when $r=b$, the steady state is expressed by

$$v=\frac{a}{b-a}\left(\frac{b}{r}-1\right). \qquad\qquad (2)$$

When b is made infinite, v assumes the limiting form a/r. In the corresponding problem for coaxal cylinders of radii a and b we have

$$v=\frac{\log b-\log r}{\log b-\log a}. \qquad\qquad (3)$$

But here there is no limiting form when b is made infinite. However great r may be, v is small when b exceeds r by only a little; but when b is great enough v may acquire any value up to unity. And since the distinction depends upon what occurs at infinity, it may evidently be extended on the one side to oval surfaces of any shape, and on the other to cylinders with any form of cross-section.

In the analogy already referred to there is correspondence between the displacements (α, β, γ) in the first case and the velocities (u, v, w) which express the motion of the viscous liquid in the second. There is also another analogy which is sometimes useful when the motion of the viscous liquid takes place in two dimensions. The *stream-function* (ψ) for this motion satisfies the same differential equation as does the transverse displacement (w') of a plane elastic plate. And a surface on which the fluid remains at rest $(\psi=0,\ d\psi/dn=0)$ corresponds to a curve along which the elastic plate is clamped.

In the light of these analogies we may conclude that, provided the square of the motion is neglected absolutely, there exists always a unique steady

motion of liquid past a solid obstacle of any form limited in all directions, which satisfies the necessary conditions both at the surface of the obstacle and at infinity, and further that the force required to hold the solid is finite. But if the obstacle be an infinite cylinder of any cross-section, no such steady motion is possible, and the force required to hold the cylinder in position continually diminishes as the motion continues.

§ 3. For further developments the simplest case is that of a material plane, coinciding with the coordinate plane $x = 0$ and moving parallel to y in a fluid originally at rest. The component velocities u, w are then zero, and the third velocity v satisfies (even though its square be not neglected) the general equation

$$\frac{dv}{dt} = \nu \frac{d^2v}{dx^2}, \qquad \text{.................................(4)}$$

in which ν, equal to μ/ρ, represents the kinematic viscosity. In § 7 of his memoir Stokes considers periodic oscillations of the plane. Thus in (4) if v be proportional to e^{int}, we have on the positive side

$$v = Ae^{int}\, e^{-x\sqrt{(in/\nu)}}. \qquad \text{.................................(5)}$$

When $x = 0$, (5) must coincide with the velocity (V) of the plane. If this be $V_n e^{int}$, we have $A = V_n$; so that in real quantities

$$v = V_n e^{-x\sqrt{(n/2\nu)}} \cos\{nt - x\sqrt{(n/2\nu)}\} \qquad \text{.................(6)}$$

corresponds with

$$V = V_n \cos nt \qquad \text{.................................(7)}$$

for the plane itself.

In order to find the tangential force ($-T_3$) exercised upon the plane we have from (5) when $x = 0$

$$\left(\frac{dv}{dx}\right)_0 = -V_n e^{int} \sqrt{(in/\nu)}, \qquad \text{.........................(8)}$$

and

$$T_3 = -\mu\,(dv/dx)_0 = \rho V_n e^{int} \sqrt{(in\nu)}$$

$$= \rho\sqrt{(\tfrac{1}{2}n\nu)}\,.\,(1+i)\,V_n e^{int} = \rho\sqrt{(\tfrac{1}{2}n\nu)}\,.\left(V + \frac{1}{n}\frac{dV}{dt}\right), \qquad \text{........(9)}$$

giving the force per unit area due to the reaction of the fluid upon one side. "The force expressed by the first of these terms tends to diminish the amplitude of the oscillations of the plane. The force expressed by the second has the same effect as increasing the inertia of the plane." It will be observed that if V_n be given, the force diminishes without limit with n.

In note B Stokes resumes the problem of § 7: instead of the motion of the plane being periodic, he supposes that the plane and fluid are initially at rest, and that the plane is then ($t = 0$) moved with a constant velocity V.

This problem depends upon one of Fourier's solutions which is easily verified*. We have

$$\frac{dv}{dx} = -\frac{V}{\sqrt{(\pi\nu t)}}\, e^{-x^2/4\nu t}, \qquad \text{.......................(10)}$$

$$v = V - \frac{2V}{\sqrt{\pi}} \int_0^{x/2\sqrt{(\nu t)}} e^{-z^2} dz. \qquad \text{.......................(11)}$$

For the reaction on the plane we require only the value of dv/dx when $x = 0$. And

$$\left(\frac{dv}{dx}\right)_0 = -\frac{V}{\sqrt{(\pi\nu t)}}. \qquad \text{.......................(12)}$$

Stokes continues† "now suppose the plane to be moved in any manner, so that its velocity at the end of the time t is $V(t)$. We may evidently obtain the result in this case by writing $V'(\tau)\,d\tau$ for V, and $t-\tau$ for t in [12], and integrating with respect to τ. We thus get

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{(\pi\nu)}} \int_{-\infty}^{t} \frac{V'(\tau)\,d\tau}{\sqrt{(t-\tau)}} = -\frac{1}{\sqrt{(\pi\nu)}} \int_0^{\infty} V'(t-t_1)\frac{dt_1}{\sqrt{t_1}}; \qquad \text{...(13)"}$$

and since $T_3 = -\mu\, dv/dx_0$, these formulæ solve the problem of finding the reaction in the general case.

There is another method by which the present problem may be treated, and a comparison leads to a transformation which we shall find useful further on. Starting from the periodic solution (8), we may generalize it by Fourier's theorem. Thus

$$\left(\frac{dv}{dx}\right)_0 = -\int_0^{\infty} V_n e^{int} \sqrt{(in/\nu)}\, dn \qquad \text{.......................(14)}$$

corresponds to

$$V(t) = \int_0^{\infty} V_n e^{int}\, dn, \qquad \text{.......................(15)}$$

where V_n is an arbitrary function of n.

Comparing (13) and (14), we see that

$$\int_0^{\infty} V_n e^{int} n^{\frac{1}{2}}\, dn = \frac{1}{\sqrt{(i\pi)}} \int_{-\infty}^{t} \frac{V'(\tau)\,d\tau}{\sqrt{(t-\tau)}}. \qquad \text{...............(16)}$$

It is easy to verify (16). If we substitute on the right for $V'(\tau)$ from (15), we get

$$\frac{1}{\sqrt{(i\pi)}} \int_{-\infty}^{t} \frac{d\tau}{\sqrt{(t-\tau)}} \int_0^{\infty} in\, V_n e^{in\tau}\, dn;$$

and taking first the integration with respect to τ,

$$\int_{-\infty}^{t} \frac{e^{in\tau}\, d\tau}{\sqrt{(t-\tau)}} = \int_0^{\infty} \frac{e^{in(t-t_1)}}{\sqrt{t_1}}\, dt_1 = \sqrt{\left(\frac{\pi}{in}\right)} \cdot e^{int},$$

when (16) follows at once.

* Compare Kelvin, *Ed. Trans.* 1862; Thomson and Tait, Appendix D.

† I have made some small changes of notation.

As a particular case of (13), let us suppose that the fluid is at rest and that the plane starts at $t=0$ with a velocity which is uniformly accelerated for a time τ_1 and afterwards remains constant. Thus from $-\infty$ to 0, $V(\tau)=0$; from 0 to τ_1, $V(\tau)=h\tau$; from τ_1 to t, where $t>\tau_1$, $V(\tau)=h\tau_1$. Thus $(0<t<\tau_1)$

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{(\pi\nu)}}\int_0^t \frac{h\,d\tau}{\sqrt{(t-\tau)}} = -\frac{2h\sqrt{t}}{\sqrt{(\pi\nu)}}; \quad \text{..............(17)}$$

and $(t>\tau_1)$

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{(\pi\nu)}}\int_0^{\tau_1} \frac{h\,d\tau}{\sqrt{(t-\tau)}} = -\frac{2h}{\sqrt{(\pi\nu)}}\{\sqrt{t}-\sqrt{(t-\tau_1)}\}. \quad \text{...(18)}$$

Expressions (17), (18), taken negatively and multiplied by μ, give the force per unit area required to propel the plane against the fluid forces acting upon *one* side. The force increases until $t=\tau_1$, that is so long as the acceleration continues. Afterwards it gradually diminishes to zero. For the differential coefficient of $\sqrt{t}-\sqrt{(t-\tau_1)}$ is negative when $t>\tau_1$; and when t is great,

$$\sqrt{t}-\sqrt{(t-\tau_1)} = \tfrac{1}{2}\tau_1 t^{-\frac{1}{2}} \quad \text{ultimately.}$$

§ 4. In like manner we may treat any problem in which the motion of the material plane is prescribed. A more difficult question arises when it is the *forces* propelling the plane that are given. Suppose, for example, that an infinitely thin vertical lamina of superficial density σ begins to fall from rest under the action of gravity when $t=0$, the fluid being also initially at rest. By (13) the equation of motion may be written

$$\frac{dV}{dt} + \frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}}\int_0^t \frac{V'(\tau)\,d\tau}{\sqrt{(t-\tau)}} = g, \quad \text{.................. ...(19)}$$

the fluid being now supposed to act on *both* sides of the lamina.

By an ingenious application of Abel's theorem Boggio has succeeded in integrating equations which include (19)*. The theorem is as follows:—If $\psi(t)$ be defined by

$$\psi(t) = \int_0^t \frac{\phi'(\tau)\,d\tau}{(t-\tau)^{\frac{1}{2}}}, \quad \text{..........................(20)}$$

then

$$\int_0^t \frac{\psi(\tau)\,d\tau}{(t-\tau)^{\frac{1}{2}}} = \pi\{\phi(t)-\phi(0)\}. \quad \text{.....................(21)}$$

For by (20), if $(t-\tau)^{\frac{1}{2}}=y$,

$$\psi(t) = 2\int_0^{\sqrt{t}} \phi'(t-y^2)\,dy;$$

* Boggio, *Rend. d. Accad. d. Lincei*, Vol. XVI. pp. 613, 730 (1907); also Basset, *Quart. Journ. of Mathematics*, No. 164, 1910, from which I first became acquainted with Boggio's work.

so that

$$\int_0^t \frac{\psi(\tau)\,d\tau}{(t-\tau)^{\frac{1}{2}}} = 2\int_0^{\sqrt{t}} \psi(t-x^2)\,dx = 4\int_0^{\sqrt{t}} dx \int_0^{\sqrt{(t-x^2)}} \phi'(t-x^2-y^2)\,dy$$

$$= 2\pi \int_0^{\sqrt{t}} \phi'(t-r^2)\,r\,dr = \pi\,\{\phi(t)-\phi(0)\},$$

where $r^2 = x^2 + y^2$.

Now, if t' be any time between 0 and t, we have, as in (19),

$$V'(t') + \frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}}\int_0^{t'} \frac{V'(\tau)\,d\tau}{\sqrt{(t'-\tau)}} = g.$$

Multiplying this by $(t-t')^{-\frac{1}{2}}dt'$ and integrating between 0 and t, we get

$$\int_0^t \frac{V'(t')\,dt'}{(t-t')^{\frac{1}{2}}} + \frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}}\int_0^t \frac{dt'}{(t-t')^{\frac{1}{2}}}\int_0^{t'} \frac{V'(\tau)\,d\tau}{(t'-\tau)^{\frac{1}{2}}} = g\int_0^t \frac{dt'}{(t-t')^{\frac{1}{2}}}. \quad \text{...(22)}$$

In (22) the first integral is the same as the integral in (19). By Abel's theorem the double integral in (22) is equal to $\pi V(t)$, since $V(0)=0$. Thus

$$\int_0^t \frac{V'(\tau)\,d\tau}{\sqrt{(t-\tau)}} + \frac{2\rho\nu^{\frac{1}{2}}\pi^{\frac{1}{2}}}{\sigma} V(t) = 2g\sqrt{t}. \quad \text{...(23)}$$

If we now eliminate the integral between (19) and (23), we obtain simply

$$\frac{dV}{dt} - \frac{4\rho^2\nu}{\sigma^2} V = g - \frac{4\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}} g\sqrt{t} \quad \text{...(24)}$$

as the differential equation governing the motion of the lamina.

This is a linear equation of the first order. Since V vanishes with t, the integral may be written

$$\frac{4\rho^2\nu V}{g\sigma^2} = e^{t'}\int_0^{t'} e^{-t}\left(1 - \frac{2\sqrt{t}}{\sqrt{\pi}}\right) dt$$

$$= \frac{2\sqrt{t'}}{\sqrt{\pi}} - 1 + \frac{2}{\sqrt{\pi}} e^{t'}\int_{\sqrt{t'}}^{\infty} e^{-x^2}\,dx, \quad \text{...(25)}$$

in which $t' = t\,.\,4\rho^2\nu/\sigma^2$. When t, or t', is great,

$$\int_{\sqrt{t'}}^{\infty} e^{-x^2}\,dx = \frac{e^{-t'}}{2\sqrt{t'}}\left(1 - \frac{1}{2t'} + \dots\right); \quad \text{...(26)}$$

so that

$$\frac{4\rho^2\nu V}{g\sigma^2} = \frac{2\sqrt{t'}}{\sqrt{\pi}} - 1 + \frac{1}{\sqrt{(\pi t')}}\left(1 - \frac{1}{2t'} + \dots\right). \quad \text{...(27)}$$

Ultimately, when t is very great,

$$V = \frac{g\sigma}{\rho}\sqrt{\left(\frac{t}{\pi\nu}\right)}. \quad \text{...(28)}$$

§ 5. The problem of the sphere moving with arbitrary velocity through a viscous fluid is of course more difficult than the corresponding problem of the plane lamina, but it has been satisfactorily solved by Boussinesq* and by Basset †. The easiest road to the result is by the application of Fourier's theorem to the periodic solution investigated by Stokes. If the velocity of the sphere at time t be $V = V_n e^{int}$, a the radius, M' the mass of the liquid displaced by the sphere, and $s = \sqrt{(n/2\nu)}$, ν being as before the kinematic viscosity, Stokes finds as the total force at time t

$$F = -M' V_n n \left\{ \left(\frac{1}{2} + \frac{9}{4sa} \right) i + \frac{9}{4sa} \left(1 + \frac{1}{sa} \right) \right\} e^{int}. \quad \text{.........(29)}$$

Thus, if

$$V = \int_0^\infty V_n e^{int} \, dn, \quad \text{.........................(30)}$$

$$F = -M' \int_0^\infty V_n n e^{int} \left\{ \left(\frac{1}{2} + \frac{9}{4sa} \right) i + \frac{9}{4sa} \left(1 + \frac{1}{sa} \right) \right\} dn. \quad \text{......(31)}$$

Of the four integrals in (31),

$$\text{the first} \quad = \tfrac{1}{2} \int_0^\infty in \, V_n e^{int} \, dn = \tfrac{1}{2} \dot{V};$$

$$\text{the fourth} = \frac{9\nu}{2a^2} \int_0^\infty V_n e^{int} \, dn = \frac{9\nu}{2a^2} V.$$

Also the second and third together give

$$\frac{9(1+i)\sqrt{(2\nu)}}{4a} \int_0^\infty V_n n^{\frac{1}{2}} e^{int} \, dn,$$

and this is the only part which could present any difficulty. We have, however, already considered this integral in connexion with the motion of a plane and its value is expressed by (16). Thus

$$F = -M' \left\{ \frac{1}{2} \frac{dV}{dt} + \frac{9\nu}{2a^2} V + \frac{9\nu^{\frac{1}{2}}}{2a\pi^{\frac{1}{2}}} \int_{-\infty}^t \frac{\dot{V}(\tau) \, d\tau}{(t-\tau)^{\frac{1}{2}}} \right\} \quad \text{.........(32)}$$

The first term depends upon the inertia of the fluid, and is the same as would be obtained by ordinary hydrodynamics when $\nu = 0$. If there is no acceleration at the moment, this term vanishes. If, further, there has been no acceleration for a long time, the third term also vanishes, and we obtain the result appropriate to a uniform motion

$$F = -\frac{9\nu M' V}{2a^2} = -6\pi a \rho \nu V = -6\pi \mu a V,$$

as in (1). The general result (32) is that of Boussinesq and Basset.

* *C. R.* t. c. p. 935 (1885); *Théorie Analytique de la Chaleur*, t. II. Paris, 1903.

† *Phil. Trans.* 1888; *Hydrodynamics*, Vol. II. chap. XXII. 1888.

As an example of (32), we may suppose (as formerly for the plane) that $V(t) = 0$ from $-\infty$ to 0; $V(t) = ht$ from 0 to τ_1; $V(t) = h\tau_1$, when $t > \tau_1$. Then if $t < \tau_1$,

$$F = -hM'\left[\frac{1}{2} + \frac{9\nu t}{2a^2} + \frac{9\nu^{\frac{1}{2}}t^{\frac{1}{2}}}{a\pi^{\frac{1}{2}}}\right]; \quad\text{.....................(33)}$$

and when $t > \tau_1$,

$$F = -hM'\left[\frac{9\nu\tau_1}{2a^2} + \frac{9\nu^{\frac{1}{2}}}{a\pi^{\frac{1}{2}}}\{\sqrt{t} - \sqrt{(t-\tau_1)}\}\right]. \quad\text{...........(34)}$$

When t is very great (34) reduces to its first term.

The more difficult problem of a sphere falling under the influence of gravity has been solved by Boggio (*loc. cit.*). In the case where the liquid and sphere are initially at rest, the solution is comparatively simple; but the analytical form of the functions is found to depend upon the ratio of densities of the sphere and liquid. This may be rather unexpected; but I am unable to follow Mr Basset in regarding it as an objection to the usual approximate equations of viscous motion.

§ 6. We will now endeavour to apply a similar method to Stokes' solution for a *cylinder* oscillating transversely in a viscous fluid. If the radius be a and the velocity V be expressed by $V = V_n e^{int}$, Stokes finds for the force

$$F = -M' in V_n e^{int}(k - ik'). \quad\text{.....................(35)}$$

In (35) M' is the mass of the fluid displaced; k and k' are certain functions of m, where $m = \frac{1}{2}a\sqrt{(n/\nu)}$, which are tabulated in his § 37. The cylinder is much less amenable to mathematical treatment than the sphere, and we shall limit ourselves to the case where, all being initially at rest, the cylinder is started with unit velocity which is afterwards steadily maintained.

The velocity V of the cylinder, which is to be zero when t is negative and unity when t is positive, may be expressed by

$$V = \tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin nt}{n}\,dn, \quad\text{.......................(36)}$$

in which the second term may be regarded as the real part of

$$\frac{1}{\pi i}\int_0^\infty \frac{e^{int}}{n}\,dn. \quad\text{.............................(37)}$$

We shall see further below, and may anticipate from Stokes' result relating to uniform motion of the cylinder, that the first term of (36) contributes nothing to F; so that we may take

$$F = -\frac{M'}{\pi}\int_0^\infty e^{int}(k - ik')\,dn,$$

corresponding to (37). Discarding the imaginary part, we get, corresponding to (36),

$$F = -\frac{M'}{\pi}\int_0^\infty (k\cos nt + k'\sin nt)\,dn. \quad\ldots\ldots\ldots\ldots\ldots(38)$$

Since k, k' are known functions of m, or (a and ν being given) of n, (38) may be calculated by quadratures for any prescribed value of t.

It appears from the tables that k, k' are positive throughout. When $m=0$, k and k' are infinite and continually diminish as m increases, until when $m=\infty$, $k=1$, $k'=0$. For small values of m the limiting forms for k, k' are

$$k = 1 + \frac{\frac{1}{4}\pi}{m^2(\log m)^2}, \qquad k' = -\frac{1}{m^2\log m}; \quad\ldots\ldots\ldots\ldots(39)$$

from which it appears that if we make n vanish in (35), while V_n is given, F comes to zero.

We now seek the limiting form when t is very great. The integrand in (38) is then rapidly oscillatory, and ultimately the integral comes to depend sensibly upon that part of the range where n is very small. And for this part we may use the approximate forms (39).

Consider, for example, the first integral in (38), from which we may omit the constant part of k. We have

$$\int_0^\infty k\cos nt\,dn = \frac{\pi}{4}\int_0^\infty \frac{\cos nt\,dn}{m^2(\log m)^2} = \frac{4\pi\nu}{a^2}\int_0^\infty \frac{\cos(4\nu a^{-2}t\,.\,x)\,dx}{x(\log x)^2}. \quad\ldots(40)$$

Writing $4\nu t/a^2 = t'$, we have to consider

$$\int_0^\infty \frac{\cos t'x\,.\,dx}{x(\log x)^2}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41)$$

In this integral the integrand is positive from $x=0$ to $x=\pi/2t'$, negative from $\pi/2t'$ to $3\pi/2t'$, and so on. For the first part of the range, if we omit the cosine,

$$\int_0^{\pi/2t'} \frac{dx}{x(\log x)^2} = \int \frac{d\log x}{(\log x)^2} = \frac{1}{\log(2t'/\pi)}; \quad\ldots\ldots\ldots\ldots\ldots(42)$$

and since the cosine is less than unity, this is an over estimate. When t' is very great, $\log(2t'/\pi)$ may be identified with $\log t'$, and to this order of approximation it appears that (41) may be represented by (42). Thus if quadratures be applied to (41), dividing the first quadrant into three parts, we have

$$\frac{\cos\pi/12}{\log 6t'/\pi} + \cos\frac{3\pi}{12}\left[\frac{1}{\log 3t'/\pi} - \frac{1}{\log 6t'/\pi}\right] + \cos\frac{5\pi}{12}\left[\frac{1}{\log 2t'/\pi} - \frac{1}{\log 3t'/\pi}\right],$$

of which the second and third terms may ultimately be neglected in comparison with the first. For example, the coefficient of $\cos(3\pi/12)$ is equal to

$$\log 2 \div \log\frac{3t'}{\pi}\,.\,\log\frac{6t'}{\pi}.$$

Proceeding in this way we see that the cosine factor may properly be identified with unity, and that the value of the integral for the first quadrant may be equated to $1/\log t'$. And for a similar reason the quadrants after the first contribute nothing of this order of magnitude. Accordingly we may take

$$\int_0^\infty k \cos nt\, dn = \frac{4\pi\nu}{a^2 \log t'}. \quad\text{.......................}(43)$$

For the other part of (38), we get in like manner

$$\int_0^\infty k' \sin nt\, dn = -\frac{8\nu}{a^2}\int_0^\infty \frac{\sin t'x\,.\,dx}{x \log x} = \frac{8\nu}{a^2}\int_0^\infty \frac{\sin x'\, dx'}{x' \log (t'/x')}. \quad\text{......}(44)$$

In the denominator of (44) it appears that ultimately we may replace $\log(t'/x')$ by $\log t'$ simply. Thus

$$\int_0^\infty k' \sin nt\, dn = \frac{4\pi\nu}{a^2 \log t'}, \quad\text{.......................}(45)$$

so that the two integrals (43), (45) are equal. We conclude that when t is great enough,

$$F = -\frac{8\nu M'}{a^2 \log t'} = -\frac{8\nu M'}{a^2 \log (4\nu t/a^2)}. \quad\text{.................}(46)$$

But a better discussion of these integrals is certainly a desideratum.

§ 7. Whatever interest the solution of the approximate equations may possess, we must never forget that the conditions under which they are applicable are very restricted, and as far as possible from being observed in many practical problems. Dynamical similarity in viscous motion requires that Va/ν be unchanged, a being the linear dimension. Thus the general form for the resistance to the uniform motion of a sphere will be

$$F = \rho\nu Va\,.f(Va/\nu), \quad\text{..........................}(47)$$

where f is an unknown function. In Stokes' solution (1) f is constant, and its validity requires that Va/ν be small*. When V is rather large, experiment shows that F is nearly proportional to V^2. In this case ν disappears. "The second power of the velocity and independence of viscosity are thus inseparably connected"†.

The general investigation for the sphere moving in any manner (in a straight line) shows that the departure from Stokes' law when the velocity is not very small must be due to the operation of the neglected terms involving the squares of the velocities; but the manner in which these act has not yet been traced. Observation shows that an essential feature in rapid fluid motion past an obstacle is the formation of a *wake* in the rear of the obstacle; but of this the solutions of the approximate equations give no hint.

* *Phil. Mag.* Vol. xxxvi. p. 354 (1893); *Scientific Papers*, Vol. iv. p. 87.

† *Phil. Mag.* Vol. xxxiv. p. 59 (1892); *Scientific Papers*, Vol. iii. p. 576.

Hydrodynamical solutions involving surfaces of discontinuity of the kind investigated by Helmholtz and Kirchhoff provide indeed for a wake, but here again there are difficulties. Behind a blade immersed transversely in a stream a region of "dead water" is indicated. The conditions of steady motion are thus satisfied; but, as Helmholtz himself pointed out, the motion thus defined is unstable. Practically the dead and live water are continually mixing; and if there be viscosity, the layer of transition rapidly assumes a finite width independently of the instability. One important consequence is the development of a suction on the hind surface of the lamina which contributes in no insignificant degree to the total resistance. The amount of the suction does not appear to depend much on the degree of viscosity. When the latter is small, the dragging action of the live upon the dead water extends to a greater distance behind.

§ 8. If the blade, supposed infinitely thin, be moved edgeways through the fluid, the case becomes one of "skin-friction." Towards determining the law of resistance Mr Lanchester has put forward an argument* which, even if not rigorous, at any rate throws an interesting light upon the question. Applied to the case of two dimensions in order to find the resistance F per unit length of blade, it is somewhat as follows. Considering two systems for which the velocity V of the blade is different, let n be the proportional width of corresponding strata of velocity. The momentum communicated to the wake per unit length of travel is as nV, and therefore on the whole as nV^2 per unit of time. Thus F varies as nV^2. Again, having regard to the law of viscosity and considering the strata contiguous to the blade, we see that F varies as V/n. Hence, nV^2 varies as V/n, or V varies as n^{-2}, from which it follows that F varies as $V^{3/2}$. If this be admitted, the general law of dynamical similarity requires that for the whole resistance

$$F = c\rho\nu^{\frac{1}{2}}\, l b^{\frac{1}{2}}\, V^{\frac{3}{2}}, \qquad\qquad (48)$$

where l is the length, b the width of the blade, and c a constant. Mr Lanchester gives this in the form

$$F/\rho = c\nu^{\frac{1}{2}} A^{\frac{3}{4}} V^{\frac{3}{2}}, \qquad\qquad (49)$$

where A is the area of the lamina, agreeing with (48) if l and b maintain a constant ratio.

The difficulty in the way of accepting the above argument as rigorous is that complete similarity cannot be secured so long as b is constant as has been supposed. If, as is necessary to this end, we take b proportional to n, it is bV/n, or V (and not V/n), which varies as nV^2, or bV^2. The conclusion is then simply that bV must be constant (ν being given). This is merely the usual condition of dynamical similarity, and no conclusion as to the law of velocity follows.

* *Aerodynamics*, London, 1907, § 35.

But a closer consideration will show, I think, that there is a substantial foundation for the idea at the basis of Lanchester's argument. If we suppose that the viscosity is so small that the layer of fluid affected by the passage of the blade is very small compared with the width (b) of the latter, it will appear that the communication of motion at any stage takes place much as if the blade formed part of an infinite plane moving as a whole. We know that if such a plane starts from rest with a velocity V afterwards uniformly maintained, the force acting upon it at time t is per unit of area, see (12),

$$\rho V \sqrt{(\nu/\pi t)}. \qquad \text{(50)}$$

The supposition now to be made is that we may apply this formula to the element of width dy, taking t equal to y/V, where y is the distance of the element from the leading edge. Thus

$$F = l\rho\,(\nu/\pi)^{\frac{1}{2}}\,V^{\frac{3}{2}}\int y^{-\frac{1}{2}}\,dy = 2l\rho\,(\nu/\pi)^{\frac{1}{2}}\,V^{\frac{3}{2}}\,b^{\frac{1}{2}}, \qquad \text{(51)}$$

which agrees with (48) if we take in the latter $c = 2/\sqrt{\pi}$.

The formula (51) would seem to be justified when ν is small enough, as representing a possible state of things; and, as will be seen, it affords an absolutely definite value for the resistance. There is no difficulty in extending it under similar restrictions to a lamina of any shape. If b, no longer constant, is the width of the lamina in the direction of motion at level z, we have

$$F = 2\rho\,(\nu/\pi)^{\frac{1}{2}}\,V^{\frac{3}{2}}\int b^{\frac{1}{2}}\,dz. \qquad \text{(52)}$$

It will be seen that the result is not expressible in terms of the *area* of the lamina. In (49) c is not constant, unless the lamina remains always similar in shape.

The fundamental condition as to the smallness of ν would seem to be realized in numerous practical cases; but any one who has looked over the side of a steamer will know that the motion is not usually of the kind supposed in the theory. It would appear that the theoretical motion is subject to instabilities which prevent the motion from maintaining its simply stratified character. The resistance is then doubtless more nearly as the square of the velocity and independent of the value of ν.

When in the case of bodies moving through air or water we express V, a, and ν in a consistent system of units, we find that in all ordinary cases ν/Va is so very small a quantity that it is reasonable to identify $f(\nu/Va)$ with $f(0)$. The influence of linear scale upon the character of the motion then disappears. This seems to be the explanation of a difficulty raised by Mr Lanchester (*loc. cit.* § 56).

355.

ABERRATION IN A DISPERSIVE MEDIUM.

[*Philosophical Magazine*, Vol. XXII. pp. 130—134, 1911.]

THE application of the theory of group-velocity to the case of light was discussed in an early paper* in connexion with some experimental results announced by Young and Forbes†. It is now, I believe, generally agreed that, whether the method be that of the toothed wheel or of the revolving mirror, what is determined by the experiment is not V, the wave-velocity, but U, the group-velocity, where

$$U = d(kV)/dk,$$

k being inversely as the wave-length. In a dispersive medium V and U are different.

I proceeded:—"The evidence of the terrestrial methods relating exclusively to U, we turn to consider the astronomical methods. Of these there are two, depending respectively upon aberration and upon the eclipses of Jupiter's satellites. The latter evidently gives U. The former does not depend upon observing the propagation of a peculiarity impressed upon a train of waves, and therefore has no relation to U. If we accept the usual theory of aberration as satisfactory, the result of a comparison between the coefficient found by observation and the solar parallax is V—the wave-velocity."

The above assertion that stellar aberration gives V rather than U has recently been called in question by Ehrenfest‡, and with good reason. He shows that the circumstances do not differ materially from those of the toothed wheel in Fizeau's method. The argument that he employs bears, indeed, close affinity with the method used by me in a later paper§. "The

* *Nature*, Vols. XXIV., XXV. 1881; *Scientific Papers*, Vol. I. p. 537.

† These observers concluded that blue light travels *in vacuo* 1·8 per cent. faster than red light.

‡ *Ann. d. Physik*, Bd. XXXIII. p. 1571 (1910).

§ *Nature*, Vol. XLV. p. 499 (1892); *Scientific Papers*, Vol. III. p. 542.

explanation of stellar aberration, as usually given, proceeds rather upon the basis of the corpuscular than of the wave-theory. In order to adapt it to the principles of the latter theory, Fresnel found it necessary to follow Young in assuming that the æther in any vacuous space connected with the earth (and therefore practically in the atmosphere) is undisturbed by the earth's motion of 19 miles per second. Consider, for simplicity, the case in which the direction of the star is at right angles to that of the earth's motion, and replace the telescope, which would be used in practice, by a pair of perforated screens, on which the light falls perpendicularly. We may further imagine the luminous disturbance to consist of a single plane pulse. When this reaches the anterior screen, so much of it as coincides with the momentary position of the aperture is transmitted, and the remainder is stopped. The part transmitted proceeds upon its course through the æther independently of the motion of the screens. In order, therefore, that the pulse may be transmitted by the aperture in the posterior screen, it is evident that the line joining the centres of the apertures must not be perpendicular to the screens and to the wave-front, as would be necessary in the case of rest. For, in consequence of the motion of the posterior screen in its own plane, the aperture will be carried forward during the time of passage of the light. By the amount of this motion the second aperture must be drawn backwards, in order that it may be in the place required when the light reaches it. If the velocity of light be V, and that of the earth be v, the line of apertures giving the apparent direction of the star must be directed forwards through an angle equal to v/V."

If the medium between the screens is dispersive, the question arises in what sense the velocity of light is to be taken. Evidently in the sense of the group-velocity; so that, in the previous notation, the aberration angle is v/U. But to make the argument completely satisfactory, it is necessary in this case to abandon the extreme supposition of a single pulse, replacing it by a group of waves of approximately given wave-length.

While there can remain no doubt but that Ehrenfest is justified in his criticism, it does not quite appear from the above how my original argument is met. There is indeed a peculiarity imposed upon the regular wave-motion constituting homogeneous light, but it would seem to be one imposed for the purposes of the argument rather than inherent in the nature of the case. The following analytical solution, though it does not relate directly to the case of a simply perforated screen, throws some light upon this question.

Let us suppose that homogeneous plane waves are incident upon a "screen" at $z = 0$, and that the effect of the screen is to introduce a reduction of the amplitude of vibration in a ratio which is slowly periodic both with respect to the time and to a coordinate x measured in the plane of the screen, represented by the factor $\cos m(vt - x)$. Thus, when $t = 0$, there is no effect

when $x = 0$, or a multiple of 2π; but when x is an odd multiple of π, there is a reversal of sign, equivalent to a change of phase of half a period. And the places where these particular effects occur travel along the screen with a velocity v which is supposed to be small relatively to that of light. In the absence of the screen the luminous vibration is represented by

$$\phi = \cos(nt - kz), \quad \text{.............................(1)}$$

or at the place of the screen, where $z = 0$, by

$$\phi = \cos nt \quad \text{simply.}$$

In accordance with the suppositions already made, the vibration just behind the screen will be

$$\begin{aligned}\phi &= \cos m(vt - x) \,.\, \cos nt \\ &= \tfrac{1}{2}\cos\{(n + mv)t - mx\} + \tfrac{1}{2}\cos\{(n - mv)t + mx\}; \quad \text{......(2)}\end{aligned}$$

and the question is to find what form ϕ will take at a finite distance z behind the screen.

It is not difficult to see that for this purpose we have only to introduce terms proportional to z into the arguments of the cosines. Thus, if we write

$$\phi = \tfrac{1}{2}\cos\{(n + mv)t - mx - \mu_1 z\} + \tfrac{1}{2}\cos\{(n - mv)t + mx - \mu_2 z\}, \quad \text{...(3)}$$

we may determine μ_1, μ_2 so as to satisfy in each case the general differential equation of propagation, viz.

$$\frac{d^2\phi}{dt^2} = V^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dz^2}\right). \quad \text{..........................(4)}$$

In (4) V is constant when the medium is non-dispersive; but in the contrary case V must be given different values, say V_1 and V_2, when the coefficient of t is $n + mv$ or $n - mv$. Thus

$$(n + mv)^2 = V_1^2(m^2 + m_1^2), \qquad (n - mv)^2 = V_2^2(m^2 + m_2^2). \quad \text{......(5)}$$

The coefficients μ_1, μ_2 being determined in accordance with (5), the value of ϕ in (3) satisfies all the requirements of the problem. It may also be written

$$\phi = \cos\{mvt - mx - \tfrac{1}{2}(\mu_1 - \mu_2)z\} \,.\, \cos\{nt - \tfrac{1}{2}(\mu_1 + \mu_2)z\}, \quad \text{......(6)}$$

of which the first factor, varying slowly with t, may be regarded as the amplitude of the luminous vibration.

The condition of constant amplitude at a given time is that $mx + \tfrac{1}{2}(\mu_1 - \mu_2)z$ shall remain unchanged. Thus the amplitude which is to be found at $x = 0$ on the screen prevails also behind the screen along the line

$$-x/z = \tfrac{1}{2}(\mu_1 - \mu_2)/m, \quad \text{..........................(7)}$$

so that (7) may be regarded as the angle of aberration due to v. It remains to express this angle by means of (5) in terms of the fundamental data.

When m is zero, the value of μ is n/V; and this is true approximately when m is small. Thus, from (5),

$$\mu_1 - \mu_2 = \frac{\mu_1^2 - \mu_2^2}{2n/V} = \frac{2mv}{V} + \frac{nV}{2}\left(\frac{1}{V_1^2} - \frac{1}{V_2^2}\right)$$

and

$$\frac{\mu_1 - \mu_2}{2m} = \frac{v}{V}\left\{1 + \frac{n}{2mv}\frac{V_2 - V_1}{V}\right\} \quad \text{.........................(8)}$$

with sufficient approximation.

Now in (8) the difference $V_2 - V_1$ corresponds to a change in the coefficient of t from $n + mv$ to $n - mv$. Hence, denoting the general coefficient of t by σ, of which V is a function, we have

$$V_1 - V_2 = 2mv \,.\, dV/d\sigma,$$

and (8) may be written

$$\frac{\mu_1 - \mu_2}{2m} = \frac{v}{V}\left\{1 - \frac{\sigma}{V}\frac{dV}{d\sigma}\right\}. \quad \text{.......................(9)}$$

Again,

$$V = \sigma/k, \qquad U = d\sigma/dk,$$

and thus

$$\frac{\sigma}{V}\frac{dV}{d\sigma} = k\frac{dV}{d\sigma} = 1 - \frac{\sigma}{k}\frac{dk}{d\sigma},$$

and

$$1 - \frac{\sigma}{V}\frac{dV}{d\sigma} = \frac{\sigma}{k}\frac{dk}{d\sigma} = \frac{V}{U},$$

where U is the group-velocity.

Accordingly,

$$-x/z = v/U \quad \text{.............................(10)}$$

expresses the aberration angle, as was to be expected. In the present problem the peculiarity impressed is not uniform over the wave-front, as may be supposed in discussing the effect of the toothed wheel; but it exists nevertheless, and it involves for its expression the introduction of more than one frequency, from which circumstance the group-velocity takes its origin.

A development of the present method would probably permit the solution of the problem of a series of equidistant moving apertures, or a single moving aperture. Doubtless in all cases the aberration angle would assume the value v/U.

LETTER TO PROFESSOR NERNST.

[*Conseil scientifique sous les auspices de M. Ernest Solvay, Oct.* 1911.]

Dear Prof. Nernst,

Having been honoured with an invitation to attend the Conference at Brussels, I feel that the least that I can do is to communicate my views, though I am afraid I can add but little to what has been already said upon the subject.

I wish to emphasize the difficulty mentioned in my paper of 1900* with respect to the use of generalized coordinates. The possibility of representing the state of a body by a finite number of such (short at any rate of the whole number of molecules) depends upon the assumption that a body may be treated as rigid, or incompressible, or in some other way simplified. The justification, and in many cases the sufficient justification, is that a departure from the simplified condition would involve such large amounts of potential energy as could not occur under the operation of the forces concerned. But the law of equi-partition lays it down that every mode is to have its share of kinetic energy. If we begin by supposing an elastic body to be rather stiff, the vibrations have their full share and this share cannot be diminished by increasing the stiffness. *For this purpose* the simplification fails, which is as much as to say that the method of generalized coordinates cannot be applied. The argument becomes, in fact, self-contradictory.

Perhaps this failure might be invoked in support of the views of Planck and his school that the laws of dynamics (as hitherto understood) cannot be applied to the smallest parts of bodies. But I must confess that I do not like this solution of the puzzle. Of course I have nothing to say against following out the consequences of the [quantum] theory of energy—a procedure which has already in the hands of able men led to some interesting

* *Phil. Mag.* Vol. XLIX. p. 118; *Scientific Papers*, Vol. IV. p. 451.

conclusions. But I have a difficulty in accepting it as a picture of what actually takes place.

We do well, I think, to concentrate attention upon the diatomic gaseous molecule. Under the influence of collisions the molecule freely and rapidly acquires rotation. Why does it not also acquire vibration along the line joining the two atoms? If I rightly understand, the answer of Planck is that in consideration of the stiffness of the union the amount of energy that should be acquired at each collision falls below the minimum possible and that therefore none at all is acquired—an argument which certainly sounds paradoxical. On the other hand Boltzmann and Jeans contend that it is all a question of time and that the vibrations necessary for full statistical equilibrium may be obtained only after thousands of years. The calculations of Jeans appear to show that there is nothing forced in such a view. I should like to inquire is there any definite experimental evidence against it? So far as I know, ordinary laboratory experience affords nothing decisive.

I am yours truly,

RAYLEIGH.

357.

ON THE CALCULATION OF CHLADNI'S FIGURES FOR A SQUARE PLATE.

[*Philosophical Magazine*, Vol. XXII. pp. 225–229, 1911.]

IN my book on the *Theory of Sound*, ch. X. (1st ed. 1877, 2nd ed. 1894) I had to speak of the problem of the vibrations of a rectangular plate, whose edges are free, as being one of great difficulty, which had for the most part resisted attack. An exception could be made of the case in which μ (the ratio of lateral contraction to longitudinal elongation) might be regarded as evanescent. It was shown that a rectangular plate could then vibrate after the same law as obtains for a simple bar, and by superposition some of the simpler Chladni's figures for a square plate were deduced. For glass and metal the value of μ is about $\frac{1}{4}$, so that for such plates as are usually experimented on the results could be considered only as rather rough approximations.

I wish to call attention to a remarkable memoir by W. Ritz* in which, somewhat on the above lines, is developed with great skill what may be regarded as a practically complete solution of the problem of Chladni's figures on square plates. It is shown that to within a few per cent. all the proper tones of the plate may be expressed by the formulæ

$$w_{mn} = u_m(x)\,u_n(y) + u_m(y)\,u_n(x),$$

$$w'_{mn} = u_m(x)\,u_n(y) - u_m(y)\,u_n(x),$$

the functions u being those proper to a free bar vibrating transversely. The coordinate axes are drawn through the centre parallel to the sides of the square. The first function of the series $u_0(x)$ is constant; the second $u_1(x) = x$. const.; $u_2(x)$ is thus the fundamental vibration in the usual sense, with two nodes, and so on. Ritz rather implies that I had overlooked the

* "Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern," *Annalen der Physik*, Bd. XXVIII. S. 737 (1909). The early death of the talented author must be accounted a severe loss to Mathematical Physics.

necessity of the first two terms in the expression of an arbitrary function. It would have been better to have mentioned them explicitly; but I do not think any reader of my book could have been misled. In § 168 the inclusion of *all** particular solutions is postulated, and in § 175 a reference is made to zero values of the frequency.

For the gravest tone of a square plate the coordinate axes are nodal, and Ritz finds as the result of successive approximations

$$\begin{aligned} w = u_1 v_1 &+ \cdot 0394\,(u_1 v_3 + v_1 u_3) \\ &- \cdot 0040\,u_3 v_3 - \cdot 0034\,(u_1 v_5 + u_5 v_1) \\ &+ \cdot 0011\,(u_3 v_5 + u_5 v_3) - \cdot 0019\,u_5 v_5\,; \end{aligned}$$

in which u stands for $u(x)$ and v for $u(y)$. The leading term $u_1 v_1$, or xy, is the same as that which I had used (§ 228) as a rough approximation on which to found a calculation of pitch.

As has been said, the general method of approximation is very skilfully applied, but I am surprised that Ritz should have regarded the method itself as new. An integral involving an unknown arbitrary function is to be made a minimum. The unknown function can be represented by a series of known functions with arbitrary coefficients—accurately if the series be continued to infinity, and approximately by a few terms. When the number of coefficients, also called generalized coordinates, is finite, they are of course to be determined by ordinary methods so as to make the integral a minimum. It was in this way that I found the correction for the open end of an organ-pipe†, using a series with two terms to express the velocity at the mouth. The calculation was further elaborated in *Theory of Sound*, Vol. II. Appendix A. I had supposed that this treatise abounded in applications of the method in question, see §§ 88, 89, 90, 91, 182, 209, 210, 265; but perhaps the most explicit formulation of it is in a more recent paper‡, where it takes almost exactly the shape employed by Ritz. From the title it will be seen that I hardly expected the method to be so successful as Ritz made it in the case of higher modes of vibration.

Being upon the subject I will take the opportunity of showing how the gravest mode of a square plate may be treated precisely upon the lines of the paper referred to. The potential energy of bending per unit area has the expression

$$V = \frac{qh^3}{3(1-\mu^2)}\left[(\nabla^2 w)^2 + 2(1-\mu)\left\{\left(\frac{d^2 w}{dx\,dy}\right)^2 - \frac{d^2 w}{dx^2}\frac{d^2 w}{dy^2}\right\}\right], \quad \ldots\ldots(1)$$

* Italics in original.

† *Phil. Trans.* Vol. CLXI. (1870); *Scientific Papers*, Vol. I. p. 57.

‡ "On the Calculation of the Frequency of Vibration of a System in its Gravest Mode, with an Example from Hydrodynamics," *Phil. Mag.* Vol. XLVII. p. 556 (1899); *Scientific Papers*, Vol. IV. p. 407.

in which q is Young's modulus, and $2h$ the thickness of the plate (§ 214). Also for the kinetic energy per unit area we have

$$T = \rho h \dot{w}^2, \qquad \text{.................................}(2)$$

ρ being the volume-density. From the symmetries of the case w must be an odd function of x and an odd function of y, and it must also be symmetrical between x and y. Thus we may take

$$w = q_1 xy + q_2 xy\,(x^2 + y^2) + q_3 xy\,(x^4 + y^4) + q_4 x^3 y^3 + \ldots \qquad \text{......}(3)$$

In the actual calculation only the two first terms will be employed.

Expressions (1) and (2) are to be integrated over the square; but it will suffice to include only the first quadrant, so that if we take the side of the square as equal to 2, the limits for x and y are 0 and 1. We find

$$\iint (\nabla^2 w)^2\, dx\, dy = 16 q_2^2, \qquad \text{.........................}(4)$$

$$\iint \left\{ \left(\frac{d^2 w}{dx\,dy} \right)^2 - \frac{d^2 w}{dx^2} \frac{d^2 w}{dy^2} \right\} dx\, dy = q_1^2 + 4 q_1 q_2 + \tfrac{8}{5} q_2^2. \qquad \text{........}(5)$$

Thus, if we set

$$V = \frac{4 q h^3}{3\,(1 + \mu)}\, V', \qquad \text{.........................}(6)$$

we have

$$V' = \tfrac{1}{2} q_1^2 + 2 q_1 q_2 + \tfrac{4}{5} q_2^2 + \frac{4 q_2^2}{1 - \mu}. \qquad \text{....................}(7)$$

In like manner, if

$$T = \frac{2 \rho h}{9}\, T', \qquad \text{..............................}(8)$$

$$T' = \tfrac{1}{2} \dot{q}_1^2 + \tfrac{6}{5} \dot{q}_1 \dot{q}_2 + \dot{q}_2^2 \left(\tfrac{3}{7} + \tfrac{9}{25} \right). \qquad \text{....................}(9)$$

When we neglect q_2 and suppose that q_1 varies as $\cos pt$, these expressions give

$$p^2 = \frac{6 q h^2}{\rho\,(1 + \mu)} = \frac{96 q h^2}{\rho\,(1 + \mu)\, a^4}, \qquad \text{......................}(10)$$

if we introduce a as the length of the side of the square. This is the value found in *Theory of Sound*, § 228, equivalent to Ritz's first approximation.

In proceeding to a second approximation we may omit the factors already accounted for in (10). Expressions (7), (9) are of the standard form if we take

$$A = 1, \qquad B = 2, \qquad C = \frac{8}{5} + \frac{8}{1 - \mu},$$

$$L = 1, \qquad M = \frac{6}{5}, \qquad N = \frac{6}{7} + \frac{18}{25};$$

and Lagrange's equations are

$$\left.\begin{aligned}(A-p^2L)\,q_1+(B-p^2M)\,q_2=0,\\(B-p^2M)\,q_1+(C-p^2N)\,q_2=0,\end{aligned}\right\} \quad\text{..................(11)}$$

while the equation for p^2 is the quadratic

$$p^4(LN-M^2)+p^2(2MB-LC-NA)+AC-B^2=0. \quad\text{.......(12)}$$

For the numerical calculations we will suppose, following Ritz, that $\mu=\cdot 225$, making $C=11\cdot 9226$. Thus

$$LN-M^2=\cdot 13714, \qquad AC-B^2=7\cdot 9226,$$
$$2MB-LC-NA=-2\times 4\cdot 3498.$$

The smaller root of the quadratic as calculated by the usual formula is ·9239, in place of the 1 of the first approximation; but the process is not arithmetically advantageous. If we substitute this value in the first term of the quadratic, and determine p^2 from the resulting simple equation, we get the confirmed and corrected value $p^2=\cdot 9241$. Restoring the omitted factors, we have finally as the result of the second approximation

$$p^2=\frac{96qh^2\times\cdot 9241}{\rho(1+\mu)a^4}, \quad\text{..........................(13)}$$

in which $\mu=\cdot 225$.

The value thus obtained is not so low, and therefore not so good, as that derived by Ritz from the series of u-functions. One of the advantages of the latter is that, being *normal* functions for the simple bar, they allow T to be expressed as a sum of squares of the generalized coordinates q_1, &c. As a consequence, p^2 appears only in the diagonal terms of the system of equations analogous to (11).

From (11) we find further

$$q_2/q_1=-\cdot 0852,$$

so that for the approximate form of w corresponding to the gravest pitch we may take

$$w=xy-\cdot 0852\,xy\,(x^2+y^2), \quad\text{......................(14)}$$

in which the side of the square is supposed equal to 2.

358.

PROBLEMS IN THE CONDUCTION OF HEAT.

[*Philosophical Magazine*, Vol. XXII. pp. 381—396, 1911.]

THE general equation for the conduction of heat in a uniform medium may be written

$$\frac{dv}{dt} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \nabla^2 v, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

v representing temperature. The coefficient (ν) denoting diffusibility is omitted for brevity on the right-hand of (1). It can always be restored by consideration of "dimensions."

Kelvin* has shown how to build up a variety of special solutions, applicable to an infinite medium, on the basis of Fourier's solution for a point-source. A few examples are quoted almost in Kelvin's words:

I. Instantaneous simple point-source; a quantity Q of heat suddenly generated at the point (0, 0, 0) at time $t=0$, and left to diffuse through an infinite homogeneous solid:

$$v = \frac{Qe^{-r^2/4t}}{8\pi^{3/2}\, t^{3/2}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $r^2 = x^2 + y^2 + z^2$. [The thermal capacity is supposed to be unity.] Verify that

$$\iiint_{-\infty}^{+\infty} v\, dx\, dy\, dz = 4\pi \int_0^\infty vr^2\, dr = Q\,;$$

and that $v=0$ when $t=0$; unless also $x=0$, $y=0$, $z=0$. *Every other solution is obtainable from this by summation.*

II. Constant simple point-source, rate q:

$$v \left[= q \int_0^\infty dt \frac{e^{-r^2/4t}}{8\pi^{3/2}\, t^{3/2}} \right] = \frac{q}{4\pi r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The formula within the brackets shows how this obvious solution is derivable from (2).

* "Compendium of Fourier Mathematics, &c.," *Enc. Brit.* 1880; *Collected Papers*, Vol. II. p. 44.

III. Continued point-source; rate per unit of time at time t, an arbitrary function, $f(t)$:

$$v=\int_0^\infty d\chi f(t-\chi)\frac{e^{-r^2/4\chi}}{8\pi^{3/2}\chi^{3/2}}. \qquad \text{.......................(4)}$$

IV. Time-periodic simple point-source, rate per unit of time at time t, $q\sin 2nt$:

$$v=\frac{q}{4\pi r}e^{-\sqrt{n}.r}\sin\left[2nt-n^{\frac{1}{2}}.r\right]. \qquad \text{....................(5)}$$

Verify that v satisfies (1); also that $-4\pi r^2 dv/dr=q\sin 2nt$, where $r=0$.

V. Instantaneous spherical surface-source; a quantity Q suddenly generated over a spherical surface of radius a, and left to diffuse outwards and inwards:

$$v=Q\frac{e^{-(r-a)^2/4t}-e^{-(r+a)^2/4t}}{8\pi^{3/2}art^{1/2}}. \qquad \text{.......................(6)}$$

To prove this most easily, verify that it satisfies (1); and further verify that

$$4\pi\int_0^\infty vr^2\,dr=Q;$$

and that $v=0$ when $t=0$, unless also $r=a$. Remark that (6) becomes identical with (2) when $a=0$; remark further that (6) is obtainable from (2) by integration over the spherical surface.

VI. Constant spherical surface-source; rate per unit of time for the whole surface, q:

$$v\left[=q\int_0^\infty dt\,\frac{e^{-(r-a)^2/4t}-e^{-(r+a)^2/4t}}{8\pi^{3/2}art^{1/2}}\right]$$
$$=q/4\pi r \quad (r>a) \qquad =q/4\pi a \quad (r<a).$$

The formula within the brackets shows how this obvious solution is derivable from (6).

VII. Fourier's "Linear Motion of Heat"; instantaneous plane-source; quantity per unit surface, σ:

$$v=\frac{\sigma e^{-x^2/4t}}{2\pi^{1/2}t^{1/2}}. \qquad \text{..............................(7)}$$

Verify that this satisfies (1) for the case of v independent of y and z, and that

$$\int_{-\infty}^{+\infty}v\,dx=\sigma.$$

Remark that (7) is obtainable from (6) by putting $Q/4\pi a^2=\sigma$, and $a=\infty$; or directly from (2) by integration over the plane.

In Kelvin's summary linear sources are passed over. If an instantaneous source be uniformly distributed along the axis of z, so that the rate per unit length is q, we obtain at once by integration from (2)

$$v = \int_{-\infty}^{+\infty} \frac{q\,dz\,e^{-(z^2+x^2+y^2)/4t}}{8\pi^{3/2}t^{3/2}} = \frac{q\,e^{-(x^2+y^2)/4t}}{4\pi t}. \quad\text{..................(8)}$$

From this we may deduce the effect of an instantaneous source uniformly distributed over a circular *cylinder* whose axis is parallel to z, the superficial density being σ. Considering the cross-section through Q the point where v is to be estimated, let O be the centre and a the radius of the circle. Then if P be a point on the circle, $OP = a$, $OQ = r$, $PQ = \rho$, $\angle POQ = \theta$; and

$$\rho^2 = a^2 + r^2 - 2ar\cos\theta,$$

so that
$$v = \int_0^{2\pi} \frac{\sigma a\,d\theta\,e^{-\rho^2/4t}}{4\pi t} = \frac{\sigma a}{2t}\,e^{-\frac{r^2+a^2}{4t}}\,I_0\left(\frac{ra}{2t}\right), \quad\text{..............(9)}$$

$I_0(x)$, equal to $J_0(ix)$, being the function usually so denoted. From (9) we fall back on (8) if we put $a = 0$, $2\pi a\sigma = q$. It holds good whether r be greater or less than a.

When x is very great and positive,

$$I_0(x) = \frac{e^x}{\sqrt{(2\pi x)}}, \quad\text{..........................(10)}$$

so that for very small values of t (9) assumes the form

$$v = \frac{\sigma a}{2\sqrt{(\pi r a t)}}\,e^{-\frac{(r-a)^2}{4t}},$$

vanishing when $t = 0$, unless $r = a$.

Again, suppose that the instantaneous source is uniformly distributed over the *circle* $\zeta = 0$, $\xi = a\cos\phi$, $\eta = a\sin\phi$, the rate per unit of arc being q, and that v is required at the point $x, 0, z$. There is evidently no loss of generality in supposing $y = 0$. We obtain at once from (2)

$$v = \int_0^{2\pi} \frac{qa\,d\phi\,e^{-r^2/4t}}{8\pi^{3/2}t^{3/2}} \quad\text{..........................(11)}$$

where
$$r^2 = (\xi - x)^2 + \eta^2 + z^2 = a^2 + x^2 + z^2 - 2ax\cos\phi.$$

Thus
$$v = \frac{qa}{4\pi^{1/2}t^{3/2}}\,e^{-\frac{a^2+x^2+z^2}{4t}}\,I_0\left(\frac{ax}{2t}\right), \quad\text{..................(12)}$$

from which if we write $q = \sigma dz$, and integrate with respect to z from $-\infty$ to $+\infty$, we may recover (9).

If in (12) we put $q = \sigma\, da$ and integrate with respect to a from 0 to ∞, we obtain a solution which must coincide with (7) when in the latter we substitute z for x. Thus

$$\int_0^\infty a\,da\,e^{-a^2/4t}\, I_0\left(\frac{ax}{2t}\right) = 2t\, e^{x^2/4t}, \qquad\text{.....................(13)}$$

a particular case of one of Weber's integrals*.

It may be worth while to consider briefly the problem of initial instantaneous sources distributed over the plane ($\zeta = 0$) in a more general manner. In rectangular coordinates the typical distribution is such that the rate per unit of area is

$$\sigma \cos l\xi \,.\, \cos m\eta. \qquad\text{..............................(14)}$$

If we assume that at x, y, z and time t, v is proportional to $\cos lx \,.\, \cos my$, the general differential equation (1) gives

$$\frac{dv}{dt} + (l^2 + m^2)\, v = \frac{d^2 v}{dz^2},$$

or

$$\frac{d}{dt}\left\{e^{(l^2+m^2)\,t}\, v\right\} = \frac{d^2}{dz^2}\left\{e^{(l^2+m^2)\,t}\, v\right\};$$

so that, as for conduction in one dimension,

$$v = A \cos lx \cos my\, e^{-(l^2+m^2)\,t}\, \frac{e^{-z^2/4t}}{\sqrt{t}}, \qquad\text{..................(15)}$$

and

$$\int_{-\infty}^{+\infty} v\,dz = 2\sqrt{\pi}\,.\, A \cos lx \cos my\, e^{-(l^2+m^2)\,t}.$$

Putting $t = 0$, and comparing with (14), we see that

$$A = \frac{\sigma}{2\sqrt{\pi}}. \qquad\text{..............................(16)}$$

By means of (2) the solution at time t may be built up from (14). In this way, by aid of the well-known integral

$$\int_{-\infty}^{+\infty} e^{-a^2x^2} \cos 2cx\, dx = \frac{\sqrt{\pi}}{a}\, e^{-c^2/a^2}, \qquad\text{....................(17)}$$

we may obtain (15) independently.

The process is of more interest in its application to polar coordinates. If we suppose that v is proportional to $\cos n\theta \,.\, J_n(kr)$,

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \frac{1}{r^2}\frac{d^2v}{d\theta^2} = -k^2 v, \qquad\text{......................(18)}$$

* Gray and Mathews' *Bessel's Functions*, p. 78, equation (160). Put $n=0$, $\lambda=0$. See also (31) below.

so that (1) gives

$$\frac{dv}{dt} + k^2 v = \frac{d^2 v}{dz^2}, \qquad \text{.......(19)}$$

and

$$v = A \cos n\theta \, J_n(kr)\, e^{-k^2 t} \frac{e^{-z^2/4t}}{\sqrt{t}}. \qquad \text{.......(20)}$$

From (20)

$$\int_{-\infty}^{+\infty} v\,dz = 2\sqrt{\pi} \,.\, A \cos n\theta \, J_n(kr)\, e^{-k^2 t}. \qquad \text{.......(21)}$$

If the initial distribution on the plane $z=0$ be per unit area

$$\sigma \cos n\theta \, J_n(kr), \qquad \text{.......(22)}$$

it follows from (21) that as before

$$A = \frac{\sigma}{2\sqrt{\pi}}. \qquad \text{.......(23)}$$

We next proceed to investigate the effect of an instantaneous source distributed over the circle for which

$$\zeta = 0, \quad \xi = a\cos\phi, \quad \eta = a \sin\phi,$$

the rate per unit length of arc being $q \cos n\phi$. From (2) at the point x, y, z

$$v = \int_0^{2\pi} \frac{q \cos n\phi \, e^{-r^2/4t}\, a\,d\phi}{8\pi^{3/2}\, t^{3/2}}, \qquad \text{.......(24)}$$

in which

$$r^2 = (\xi - x)^2 + (\eta - y)^2 + z^2 = a^2 + \rho^2 + z^2 - 2a\rho\cos(\phi - \theta),$$

if $x = \rho\cos\theta$, $y = \rho\sin\theta$. The integral that we have to consider may be written

$$\int_0^{2\pi} \cos n\phi \, e^{\rho' \cos(\phi-\theta)}\, d\phi = \int \cos n(\theta + \psi)\, e^{\rho'\cos\psi}\, d\psi$$

$$= \cos n\theta \int \cos n\psi \, e^{\rho'\cos\psi}\, d\psi - \sin n\theta \int \sin n\psi \, e^{\rho'\cos\psi}\, d\psi, \qquad \text{......(25)}$$

where $\psi = \phi - \theta$, and $\rho' = a\rho/2t$. In view of the periodic character of the integrand, the limits may be taken as $-\pi$ and $+\pi$. Accordingly

$$\int_{-\pi}^{+\pi} \cos n\psi \, e^{\rho'\cos\psi}\, d\psi = 2\int_0^{\pi} \cos n\psi \, e^{\rho'\cos\psi}\, d\psi,$$

$$\int_{-\pi}^{+\pi} \sin n\psi \, e^{\rho'\cos\psi}\, d\psi = 0;$$

and

$$\int_0^{2\pi} \cos n\phi \, e^{\rho'\cos(\phi-\theta)}\, d\phi = 2\cos n\theta \int_0^{\pi} \cos n\psi \, e^{\rho'\cos\psi}\, d\psi. \qquad \text{......(26)}$$

The integral on the right of (26) is equivalent to $\pi I_n(\rho')$, where

$$i^n I_n(\rho') = J_n(i\rho'), \qquad \text{.......(27)}$$

J_n being, as usual, the symbol of Bessel's function of order n. For, if n be *even*,

$$\int_0^\pi \cos n\psi\, e^{\rho'\cos\psi}\, d\psi = \tfrac{1}{2}\int_0^\pi \cos n\psi\, (e^{\rho'\cos\psi} + e^{-\rho'\cos\psi})\, d\psi$$

$$= \int_0^\pi \cos n\psi \cos(i\rho'\cos\psi)\, d\psi = \pi i^{-n} J_n(i\rho') = \pi I_n(\rho');$$

and, if n be *odd*,

$$\int_0^\pi \cos n\psi\, e^{\rho'\cos\psi}\, d\psi = -\tfrac{1}{2}\int_0^\pi \cos n\psi\, (e^{-\rho'\cos\psi} - e^{\rho'\cos\psi})\, d\psi$$

$$= -i\int_0^\pi \cos n\psi \sin(i\rho'\cos\psi)\, d\psi = \pi I_n(\rho').$$

In either case

$$\int_0^\pi \cos n\psi\, e^{\rho'\cos\psi}\, d\psi = \pi I_n(\rho'). \quad\text{.....................(28)}$$

Thus
$$\int_0^{2\pi} \cos n\phi\, e^{\rho'\cos(\phi-\theta)}\, d\phi = 2\pi\cos n\theta\, I_n(\rho'), \quad\text{..............(29)}$$

and (24) becomes

$$v = \frac{qa\cos n\theta}{4\pi^{1/2}t^{3/2}} I_n\left(\frac{a\rho}{2t}\right) e^{-\frac{a^2+\rho^2+z^2}{4t}}. \quad\text{.....................(30)}$$

This gives the temperature at time t and place (ρ, z) due to an initial instantaneous source distributed over the circle a.

The solution (30) may now be used to find the effect of the initial source expressed by (22). For this purpose we replace q by $\sigma\, da$, and introduce the additional factor $J_n(ka)$, subsequently integrating with respect to a between the limits 0 and ∞. Comparing the result with that expressed in (20), (23), we see that

$$\frac{\sigma\cos n\theta\, e^{-z^2/4t}}{2\sqrt{(\pi t)}}$$

is a common factor which divides out, and that there remains the identity

$$\frac{e^{-\rho^2/4t}}{2t}\int_0^\infty a\,da\, e^{-a^2/4t} J_n(ka)\, I_n\left(\frac{a\rho}{2t}\right) = J_n(k\rho)\, e^{-k^2t}. \quad\text{.......(31)}$$

This agrees with the formula given by Weber, which thus receives an interesting interpretation.

Reverting to (30), we recognize that it must satisfy the fundamental equation (1), now taking the form

$$\frac{d^2v}{dz^2} + \frac{d^2v}{d\rho^2} + \frac{1}{\rho}\frac{dv}{d\rho} + \frac{1}{\rho^2}\frac{d^2v}{d\theta^2} = \frac{dv}{dt}; \quad\text{.....................(32)}$$

and that when $t = 0$ v must vanish, unless also $z = 0$, $\rho = a$.

If we integrate (30) with respect to z between $\pm\infty$, setting $q = \sigma dz$, so that $\sigma\cos n\theta$ represents the superficial density of the instantaneous source distributed over the *cylinder* of radius a, we obtain

$$v = \frac{\sigma a \cos n\theta}{2t} I_n\left(\frac{ap}{2t}\right) e^{-\frac{a^2+\rho^2}{4t}}, \quad \text{.....................(33)}$$

which may be regarded as a generalization of (9). And it appears that (33) satisfies (32), in which the term d^2v/dz^2 may now be omitted.

In V. Kelvin gives the temperature at a distance r from the centre and at time t due to an instantaneous source uniformly distributed over a spherical surface. In deriving the result by integration from (2) it is of course simplest to divide the spherical surface into elementary circles which are symmetrically situated with respect to the line OQ joining the centre of the sphere O to the point Q where the effect is required. But if the circles be drawn round another axis OA, a comparison of results will give a definite integral.

Adapting (12), we write $a = c\sin\theta$, c being the radius of the sphere, $x = OQ\sin\theta' = r\sin\theta'$, $z = r\cos\theta' - c\cos\theta$, so that

$$v = \frac{qc\sin\theta\, e^{-(c^2+r^2)/4t}}{4\pi^{1/2}\, t^{3/2}} I_0\left(\frac{cr\sin\theta\sin\theta'}{2t}\right) e^{\frac{rc\cos\theta\cos\theta'}{2t}} \quad \text{.........(34)}$$

This has now to be integrated with respect to θ from 0 to π. Since the result must be independent of θ', we see by putting $\theta' = 0$ that

$$\int_0^\pi I_0(\rho\sin\theta\sin\theta')\, e^{\rho\cos\theta\cos\theta'}\sin\theta\, d\theta$$

$$= \int_0^\pi e^{\rho\cos\theta}\sin\theta\, d\theta = \frac{1}{\rho}(e^\rho - e^{-\rho}). \quad \text{...........(35)}$$

Using the simplified form and putting $q = \sigma c\, d\theta$, where σ is the superficial density, we obtain for the complete sphere

$$v = \frac{\sigma c}{2\pi^{\frac{1}{2}} t^{\frac{1}{2}} r}\left(e^{-\frac{(c-r)^2}{4t}} - e^{-\frac{(c+r)^2}{4t}}\right), \quad \text{.................(36)}$$

agreeing with (6) when we remember that $Q = 4\pi c^2\sigma$.

We will now consider the problem of an instantaneous source arbitrarily distributed over the surface of the sphere whose radius is c. It suffices, of course, to treat the case of a spherical harmonic distribution; and we suppose that per unit of area of the spherical surface the rate is S_n. Assuming that v is everywhere proportional to S_n, we know that v satisfies

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2v}{d\omega^2} + n(n+1)v = 0, \quad \text{.........(37)}$$

θ, ω being the usual spherical polar coordinates. Hence from (1) v as a function of r and t satisfies

$$\frac{dv}{dt} = \frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} - \frac{n(n+1)v}{r^2} = 0,$$

or
$$\frac{d(rv)}{dt} = \frac{d^2(rv)}{dr^2} - \frac{n(n+1)}{r^2}(rv) = 0. \qquad (38)$$

When $n = 0$, this reduces to the same form as applies in one dimension. For general values of n the required solution appears to be most easily found indirectly.

Let us suppose that S_n reduces to Legendre's function $P_n(\mu)$, where $\mu = \cos\theta$, and let us calculate directly from (2) the value of v at time t and at a point Q distant r from the centre of the sphere *along the axis of* μ. The exponential term is

$$e^{-\frac{r^2+c^2}{4t}} e^{\frac{rc\mu}{2t}} = e^{-\frac{r^2+c^2}{4t}} e^{\rho\mu}, \qquad (39)$$

if $\rho = rc/2t$. Now (*Theory of Sound*, § 334)

$$\int_{-1}^{+1} P_n(\mu)\, e^{i\rho\mu}\, d\mu = 2i^n \sqrt{\left(\frac{\pi}{2\rho}\right)} J_{n+\frac{1}{2}}(\rho), \qquad (40)$$

whence
$$\int_{-1}^{+1} P_n(\mu)\, e^{\rho\mu}\, d\mu = 2i^{n+\frac{1}{2}} \sqrt{\left(\frac{\pi}{2\rho}\right)} J_{n+\frac{1}{2}}(-i\rho), \qquad (41)$$

or, as it may also be written by (27),

$$= 2\sqrt{\left(\frac{\pi}{2\rho}\right)} I_{n+\frac{1}{2}}(\rho). \qquad (42)$$

Substituting in (2)

$$Q = 2\pi c^2 P_n(\mu)\, d\mu, \qquad (43)$$

we now get for the value of v at time t, and at the point for which $\rho = r$, $\mu = 1$,

$$rv = \frac{i^{n+\frac{1}{2}} c^{3/2} r^{\frac{1}{2}} e^{-(r^2+c^2)/4t}}{2t} J_{n+\frac{1}{2}}\left(-\frac{irc}{2t}\right). \qquad (44)$$

It may be verified by trial that (44) is a solution of (38). When μ is not restricted to the value unity, the only change required in (44) is the introduction of the factor $P_n(\mu)$.

When $n = 0$, $P_n(\mu) = 1$, and we fall back upon the case of *uniform* distribution. We have

$$J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x, \qquad (45)$$

or
$$J_{\frac{1}{2}}(-ix) = i^{-\frac{1}{2}} \frac{e^x - e^{-x}}{\sqrt{(2\pi x)}}. \qquad (46)$$

Using this in (44), we obtain a result in accordance with (6), in which Q, representing the integrated magnitude of the source, is equal to $4\pi c^2$ in our present reckoning.

When $n = 1$, $P_1(\mu) = \mu$, and

$$J_{3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\frac{\sin x}{x} - \cos x\right\}; \qquad \text{.................. (47)}$$

and whatever integral value n may assume $J_{n+\frac{1}{2}}$ is expressible in finite terms.

We have supposed that the rate of distribution is represented by a Legendre's function $P_n(\mu)$. In the more general case it is evident that we have merely to multiply the right-hand member of (44) by S_n, instead of P_n.

So far we have been considering instantaneous sources. As in II., the effect of *constant* sources may be deduced by integration, although the result is often more readily obtained otherwise. A comparison will, however, give the value of a definite integral. Let us apply this process to (33) representing the effect of a cylindrical source.

The required solution, being independent of t, is obtained at once from (1). We have inside the cylinder

$$v = A\rho^n \cos n\theta,$$

and outside

$$v = B\rho^{-n} \cos n\theta,$$

with $Aa^n = Ba^{-n}$. The intensity of the source is represented by the difference in the values of $dv/d\rho$ just inside and just outside the cylindrical surface. Thus

$$\sigma' \cos n\theta = n \cos n\theta \,(Ba^{-n-1} + Aa^{n-1}),$$

whence

$$Aa^n = Ba^{-n} = \sigma' a/2n,$$

$\sigma' \cos n\theta$ being the constant time rate. Accordingly, within the cylinder

$$v = \frac{\sigma' a}{2n}\left(\frac{\rho}{a}\right)^n \cos n\theta, \qquad \text{.......................... (48)}$$

and without the cylinder

$$v = \frac{\sigma' a}{2n}\left(\frac{\rho}{a}\right)^{-n} \cos n\theta. \qquad \text{.......................... (49)}$$

These values are applicable when n is any positive integer. When n is zero, there is no permanent distribution of temperature possible.

These solutions should coincide with the value obtained from (33) by putting $\sigma = \sigma' dt$ and integrating with respect to t from 0 to ∞. Or

$$\int_0^\infty \frac{dt}{t}\, I_n\left(\frac{a\rho}{2t}\right) e^{-\frac{a^2+\rho^2}{4t}} = \frac{1}{n}\left(\frac{\rho}{a}\right)^{\pm n}, \qquad \text{.................. (50)}$$

the + sign in the ambiguity being taken when $\rho < a$, and the − sign when $\rho > a$. I have not confirmed (50) independently.

In like manner we may treat a constant source distributed over a *sphere*. If the rate per unit time and per unit of area of surface be S_n, we find, as above, for inside the sphere (c)

$$v = \frac{c}{2n+1}\left(\frac{r}{c}\right)^n S_n, \quad \text{.........................(51)}$$

and outside the sphere

$$v = \frac{c}{2n+1}\left(\frac{c}{r}\right)^{n+1} S_n, \quad \text{.........................(52)}$$

and these forms are applicable to any integral n, *zero included*. Comparing with (44), we see that

$$i^{n+\frac{1}{2}}\int_0^\infty \frac{dt}{t} e^{-\frac{r^2+c^2}{4t}} J_{n+\frac{1}{2}}\left(-\frac{irc}{2t}\right) = \frac{2}{2n+1}\left(\frac{r}{c}\right)^{\pm(n+\frac{1}{2})}, \quad \text{........(53)}$$

which does not differ from (50), if in the latter we suppose n = integer + $\frac{1}{2}$.

The solution for a time-periodic simple point-source has already been quoted from Kelvin (IV.). Though derivable as a particular case from (4), it is more readily obtained from the differential equation (1) taking here the form—see (38) with $n=0$—

$$\frac{d^2(rv)}{dt} = \frac{d^2(rv)}{dr^2},$$

or if v is assumed proportional to e^{ipt},

$$d^2(rv)/dr^2 - ip\,(rv) = 0, \quad \text{.........................(54)}$$

giving

$$rv = Ae^{ipt}\, e^{-i^{\frac{1}{2}}p^{\frac{1}{2}}r}, \quad \text{.............................(55)}$$

as the symbolical solution applicable to a source situated at $r=0$. Denoting by q the magnitude of the source, as in (5), we get to determine A,

$$\left[-4\pi r^2 \frac{dv}{dr}\right]_{r=0} = q\, e^{ipt} = 4\pi A,$$

so that

$$v = \frac{q}{4\pi r} e^{ipt}\, e^{-i^{\frac{1}{2}}p^{\frac{1}{2}}r}. \quad \text{...........................(56)}$$

If from (56) we discard the imaginary part, we have

$$v = \frac{q}{4\pi r} e^{-r\sqrt{(p/2)}} \cos\{pt - r\sqrt{(p/2)}\}, \quad \text{.................(57)}$$

corresponding to the source $q \cos pt$.

From (56) it is possible to build up by integration solutions relating to various distributions of periodic sources over lines or surfaces, but an independent treatment is usually simpler. We will, however, write down the integral corresponding to a uniform linear source coincident with the axis of z. If $\rho^2 = x^2 + y^2$, $r^2 = z^2 + \rho^2$, and (ρ being constant) $r\,dr = z\,dz$. Thus putting in (56) $q = q_1\,dz$, we get

$$v = \frac{q_1 e^{ipt}}{2\pi}\int_\rho^\infty \frac{e^{-r\sqrt{(ip)}}\,dr}{\sqrt{(r^2-\rho^2)}}. \quad \text{.......................(58)}$$

In considering the effect of periodic sources distributed over a plane xy, we may suppose

$$v \propto \cos lx \,.\, \cos my, \quad \text{.........................(59)}$$

or again

$$v \propto J_n(kr) \,.\, \cos n\theta, \quad \text{.........................(60)}$$

where $r^2 = x^2 + y^2$. In either case if we write $l^2 + m^2 = k^2$, and assume v proportional to e^{ipt}, (1) gives

$$d^2v/dz^2 = (k^2 + ip)\, v. \quad \text{.........................(61)}$$

Thus, if

$$k^2 + ip = R\,(\cos\alpha + i\sin\alpha), \quad \text{.........................(62)}$$

$$v = Ae^{-\sqrt{R}\,.\,(\cos\frac{1}{2}\alpha + i\sin\frac{1}{2}\alpha)\,z}\, e^{ipt}, \quad \text{.........................(63)}$$

where A includes the factors (59) or (60). If the value of v be given on the plane $z = 0$, that of A follows at once. If the magnitude of the source be given, A is to be found from the value of dv/dz when $z = 0$.

The simplest case is of course that where $k = 0$. If Ve^{ipt} be the value of v when $z = 0$, we find

$$v = Ve^{ipt}\, e^{-z\sqrt{(ip)}}\,; \quad \text{.........................(64)}$$

or when realized

$$v = Ve^{-z\sqrt{(p/2)}} \cos\{pt - z\sqrt{(p/2)}\}, \quad \text{.................(65)}$$

corresponding to

$$v = V\cos pt \qquad \text{when } z = 0.$$

From (64)

$$-\left(\frac{dv}{dz}\right)_0 = \sqrt{(ip)}\,.\, Ve^{ipt} = \tfrac{1}{2}\sigma\, e^{ipt}, \quad \text{.................(66)}$$

if σ be the source per unit of area of the plane regarded as operative in a medium indefinitely extended in both directions. Thus in terms of σ,

$$v = \frac{\sigma}{2\sqrt{p}}\, e^{i(pt - \frac{1}{4}\pi)}\, e^{-z\sqrt{(ip)}}, \quad \text{.........................(67)}$$

or in real form

$$v = \frac{\sigma}{2\sqrt{p}}\, e^{-z\sqrt{(p/2)}} \cos\{pt - \tfrac{1}{4}\pi - z\sqrt{(p/2)}\}, \quad \text{..............(68)}$$

corresponding to the uniform source $\sigma\cos pt$.

In the above formulæ z is supposed to be positive. On the other side of the source, where z itself is negative, the signs must be changed so that the terms containing z may remain negative in character.

When periodic sources are distributed over the surface of a *sphere* (radius $= c$), we may suppose that v is proportional to the spherical surface harmonic S_n. As a function of r and t, v is then subject to (38); and when we introduce the further supposition that as dependent on t, v is proportional to e^{ipt}, we have

$$\frac{d^2(rv)}{dr^2} - \frac{n(n+1)}{r^2}(rv) - ip\,(rv) = 0. \quad \text{..............(69)}$$

When $n=0$, that is in the case of symmetry round the pole, this equation takes the same form as for one dimension; but we have to distinguish between the inside and the outside of the sphere.

On the inside the constants must be so chosen that v remains finite at the pole ($r=0$). Hence

$$rv = A\,e^{ipt}\,(e^{r\sqrt{(ip)}} - e^{-r\sqrt{(ip)}}), \qquad\qquad (70)$$

or in real form

$$rv = Ae^{r\sqrt{(p/2)}}\cos\{pt + r\sqrt{(p/2)}\} - Ae^{-r\sqrt{(p/2)}}\cos\{pt - r\sqrt{(p/2)}\}. \qquad (71)$$

Outside the sphere the condition is that rv must vanish at infinity. In this case

$$rv = B\,e^{ipt}\,e^{-r\sqrt{(ip)}}, \qquad\qquad (72)$$

or in real form

$$rv = Be^{-r\sqrt{(p/2)}}\cos\{pt - r\sqrt{(p/2)}\}. \qquad\qquad (73)$$

When n is not zero, the solution of (69) may be obtained as in Stokes' treatment of the corresponding acoustical problem (*Theory of Sound*, ch. XVII). Writing $r\sqrt{(ip)} = z$, and assuming

$$rv = Ae^{z} + Be^{-z}, \qquad\qquad (74)$$

where A and B are functions of z, we find for B

$$\frac{d^2B}{dz^2} - 2\frac{dB}{dz} - \frac{n(n+1)}{z^2}B = 0. \qquad\qquad (75)$$

The solution is

$$B = B_0 f_n(z), \qquad\qquad (76)$$

where B_0 is independent of z and

$$f_n(z) = 1 + \frac{n(n+1)}{2\,.\,z} + \frac{(n-1)\,n\,(n+1)\,(n+2)}{2\,.\,4\,.\,z^2} + \ldots, \qquad (77)$$

as may be verified by substitution. Since n is supposed integral, the series (77) terminates. For example, if $n=1$, it reduces to the first two terms.

The solution appropriate to the exterior is thus

$$rv = B_0 S_n e^{ipt}\,e^{-r\sqrt{(ip)}}\,f_n(i^{\frac{1}{2}}p^{\frac{1}{2}}r). \qquad\qquad (78)$$

For the interior we have

$$rv = A_0 S_n e^{ipt}\,\{e^{-r\sqrt{(ip)}}\,f_n(i^{\frac{1}{2}}p^{\frac{1}{2}}r) - e^{r\sqrt{(ip)}}\,f_n(-i^{\frac{1}{2}}p^{\frac{1}{2}}r)\}, \qquad (79)$$

which may also be expressed by a Bessel's function of order $n+\frac{1}{2}$.

In like manner we may treat the problem in two dimensions, where everything may be expressed by the polar coordinates r, θ. It suffices to consider the terms in $\cos n\theta$, where n is an integer. The differential equation analogous to (69) is now

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{n^2}{r^2}v = ipv, \qquad\qquad (80)$$

which, if we take $r\sqrt{(ip)} = z$, as before, may be written

$$\frac{d^2(z^{\frac{1}{2}}v)}{dz^2} - \frac{(n-\frac{1}{2})(n+\frac{1}{2})}{z^2}(z^{\frac{1}{2}}v) = z^{\frac{1}{2}}v, \quad\text{.................(81)}$$

and is of the same form as (69) when in the latter $n - \frac{1}{2}$ is written for n.

As appears at once from (80), the solution for the interior of the cylinder may be expressed

$$v = A\cos n\theta\, e^{ipt} J_n(i^{1/2}p^{1/2}r), \quad\text{....................(82)}$$

J_n being as usual the Bessel's function of the nth order.

For the exterior we have from (81)

$$r^{\frac{1}{2}}v = B\cos n\theta\, e^{ipt} e^{-r\sqrt{(ip)}} f_{n-\frac{1}{2}}(i^{\frac{1}{2}}p^{\frac{1}{2}}r), \quad\text{..............(83)}$$

where

$$f_{n-\frac{1}{2}}(z) = 1 + \frac{4n^2-1^2}{1\,.\,8z} + \frac{(4n^2-1^2)(4n^2-3^2)}{1\,.\,2\,.\,(8z)^2}$$
$$+ \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{1\,.\,2\,.\,3\,.\,(8z)^3} + \dots \quad\text{........(84)}$$

The series (84), unlike (77), does not terminate. It is ultimately divergent, but may be employed for computation when z is moderately great.

In these periodic solutions the sources distributed over the plane, sphere, or cylinder are supposed to have been in operation for so long a time that any antecedent distribution of temperature throughout the medium is without influence. By Fourier's theorem this procedure may be generalized. Whatever be the character of the sources with respect to time, it may be resolved into simple periodic terms; and if the character be known through the whole of past time, the solution so obtained is unambiguous. The same conclusion follows if, instead of the magnitude of the sources, the temperature at the surfaces in question be known through past time.

An important particular case is when the character of the function is such that the superficial value, having been constant (zero) for an infinite time, is suddenly raised to another value, say unity, and so maintained. The Fourier expression for such a function is

$$\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin pt}{p}\,dp, \quad\text{.........................(85)}$$

the definite integral being independent of the arithmetical value of t, but changing sign when t passes through 0; or, on the understanding that only the real part is to be retained,

$$\frac{1}{2} + \frac{1}{i\pi}\int_0^\infty \frac{e^{ipt}}{p}\,dp. \quad\text{...........................(86)}$$

We may apply this at once to the case of the plane $z = 0$ which has been at 0 temperature from $t = -\infty$ to $t = 0$, and at temperature 1 from $t = 0$ to $t = \infty$. By (64)

$$v = \tfrac{1}{2} + \frac{1}{i\pi}\int_0^\infty \frac{e^{ipt - z\sqrt{(ip)}}}{p}\, dp. \quad \text{.......................(87)}$$

By the methods of complex integration this solution may be transformed into Fourier's, viz.

$$\frac{dv}{dz} = -\frac{1}{\sqrt{(\pi t)}}\, e^{-z^2/4t}, \quad \text{.................................(88)}$$

$$v = 1 - \frac{2}{\sqrt{\pi}}\int_0^{z/2\sqrt{t}} e^{-\alpha^2} d\alpha, \quad \text{.......................(89)}$$

which are, however, more readily obtained otherwise.

In the case of a cylinder ($r = c$) whose surface has been at 0 up to $t = 0$ and afterwards at $v = 1$, we have from (83) with $n = 0$

$$v = \tfrac{1}{2} + \frac{c^{\frac{1}{2}}}{i\pi r^{\frac{1}{2}}}\int_0^\infty e^{ipt + (c-r)\sqrt{(ip)}}\, \frac{f_{-\frac{1}{2}}(i^{\frac{1}{2}} p^{\frac{1}{2}} r)\, dp}{f_{-\frac{1}{2}}(i^{\frac{1}{2}} p^{\frac{1}{2}} c)\; p}, \quad \text{............(90)}$$

of which only the real part is to be retained. This applies to the region outside the cylinder.

It may be observed that when t is negative (87) must vanish for positive z and (90) for $r > c$.

359.

ON THE GENERAL PROBLEM OF PHOTOGRAPHIC REPRODUCTION, WITH SUGGESTIONS FOR ENHANCING GRADATION ORIGINALLY INVISIBLE.

[*Philosophical Magazine*, Vol. XXII. pp. 734—740, 1911.]

IN copying a subject by photography the procedure usually involves two distinct steps. The first yields a so-called *negative*, from which, by the same or another process, a second operation gives the desired *positive*. Since ordinary photography affords pictures in monochrome, the reproduction can be complete only when the original is of the same colour. We may suppose, for simplicity of statement, that the original is itself a transparency, *e.g.* a lantern-slide.

The character of the original is regarded as given by specifying the transparency (t) at every point, *i.e.* the ratio of light transmitted to light incident. But here an ambiguity should be noticed. It may be a question of the place at which the transmitted light is observed. When light penetrates a stained glass, or a layer of coloured liquid contained in a tank, the direction of propagation is unaltered. If the incident rays are normal, so also are the rays transmitted. The action of the photographic image, constituted by an imperfectly aggregated deposit, differs somewhat. Rays incident normally are more or less diffused after transmission. The effective transparency in the half-tones of a negative used for contact printing may thus be sensibly greater than when a camera and lens is employed. In the first case all the transmitted light is effective; in the second most of that diffused through a finite angle fails to reach the lens*. In defining t—the transparency at any place—account must in strictness be taken of the manner in which the picture is to be viewed. There is also another point to be considered. The transparency may not be the same for different kinds

* In the extreme case a negative seen against a dark background and lighted obliquely from behind may even appear as a positive.

of light. We must suppose either that one kind of light only is employed, or else that t is the same for all the kinds that need to be regarded. The actual values of t may be supposed to range from 0, representing complete opacity, to 1, representing complete transparency.

As the first step is the production of a negative, the question naturally suggests itself whether we can define the ideal character of such a negative. Attempts have not been wanting; but when we reflect that the negative is only a means to an end, we recognize that no answer can be given without reference to the process in which the negative is to be employed to produce the positive. In practice this process (of printing) is usually different from that by which the negative was itself made; but for simplicity we shall suppose that the same process is employed in both operations. This requirement of identity of procedure in the two cases is to be construed strictly, extending, for example, to duration of development and degree of intensification, if any. Also we shall suppose for the present that the *exposure* is the same. In strictness this should be understood to require that both the intensity of the incident light and the time of its operation be maintained; but since between wide limits the effect is known to depend only upon the product of these quantities, we may be content to regard exposure as defined by a single quantity, viz. *intensity of light* × *time*.

Under these restrictions the transparency t' at any point of the negative is a definite function of the transparency t at the corresponding point of the original, so that we may write

$$t' = f(t), \quad \text{.................................(1)}$$

f depending upon the photographic procedure and being usually such that as t increases from 0 to 1, t' decreases continually. When the operation is repeated upon the negative, the transparency t'' at the corresponding part of the positive is given by

$$t'' = f(t'). \quad \text{.................................(2)}$$

Complete reproduction may be considered to demand that at every point $t'' = t$. Equation (2) then expresses that t must be the same function of t' that t' is of t. Or, if the relation between t and t' be written in the form

$$F(t, t') = 0, \quad \text{.................................(3)}$$

F must be a *symmetrical* function of the two variables. If we regard t, t' as the rectangular coordinates of a point, (3) expresses the relationship by a *curve* which is to be symmetrical with respect to the bisecting line $t' = t$.

So far no particular form of f, or F, is demanded; no particular kind of negative is indicated as ideal. But certain simple cases call for notice. Among these is

$$t + t' = 1, \quad \text{.................................(4)}$$

which obviously satisfies the condition of symmetry. The representative curve is a straight line, equally inclined to the axes. According to (4), when $t=0$, $t'=1$. This requirement is usually satisfied in photography, being known as freedom from fog—no photographic action where no light has fallen. But the complementary relation $t'=0$ when $t=1$ is only satisfied approximately. The relation between negative and positive expressed in (4) admits of simple illustration. If both be projected upon a screen from independent lanterns of equal luminous intensity, so that the images fit, the pictures obliterate one another, and there results a field of uniform intensity.

Another simple form, giving the same limiting values as (4), is

$$t^2+t'^2=1\,; \quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and of course any number of others may be suggested.

According to Fechner's law, which represents the facts fairly well, the visibility of the difference between t and $t+dt$ is proportional to dt/t. The gradation in the negative, constituted in agreement with (4), is thus quite different from that of the positive. When t is small, large differences in the positive may be invisible in the negative, and *vice versâ* when t approaches unity. And the want of correspondence in gradation is aggravated if we substitute (5) for (4). All this is of course consistent with complete final reproduction, the differences which are magnified in the first operation being correspondingly attenuated in the second.

If we impose the condition that the gradation in the negative shall agree with that in the positive, we have

$$dt/t=-dt'/t', \quad\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

whence

$$t\,.\,t'=C, \quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where C is a constant. This relation does not fully meet the other requirements of the case. Since t' cannot exceed unity, t cannot be less than C. However, by taking C small enough, a sufficient approximation may be attained. It will be remarked that according to (7) the negative and positive obliterate one another when superposed in such a manner that light passes through them in succession— a combination of course entirely different from that considered in connexion with (4). This equality of gradation (within certain limits) may perhaps be considered a claim for (7) to represent the ideal negative; on the other hand, the *word* accords better with definition (4).

It will be remembered that hitherto we have assumed the exposure to be the same in the two operations, viz. in producing the negative and in copying from it. The restriction is somewhat arbitrary, and it is natural to inquire whether it can be removed. One might suppose that the removal would allow a greater latitude in the relationship between t and t'; but a closer scrutiny seems to show that this is not the case.

The effect of varying the exposure (e) is the same as of an inverse alteration in the transparency; it is the product et with which we really have to do. This refers to the first operation; in the second, t'' is dependent in like manner upon $e't'$. For simplicity and without loss of generality we may suppose that $e = 1$; also that $e'/e = m$, where m is a numerical quantity greater or less than unity. The equations which replace (1) and (2) are now

$$t' = f(t), \qquad t = t'' = f(mt'); \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

and we assume that f is such that it decreases continually as its argument increases. This excludes what is called in photography *solarization*.

We observe that if t, lying between 0 and 1, anywhere makes $t' = t$, then m must be taken to be unity. For in the case supposed

$$t = f(t) = f(mt);$$

and this in accordance with the assumed character of f cannot be true, unless $m = 1$. Indeed without analytical formulation it is evident that since the transparency is not altered in the negative, it will require the same exposure to obtain it in the second operation as that by which it was produced in the first. Hence, if anywhere $t' = t$, the exposures must be the same.

It remains to show that there is no escape from a local equality of t and t'. When $t = 0$, $t' = 1$, or (if there be fog) some smaller positive quantity. As t increases from 0 to 1, t' continually decreases, and must therefore pass t at some point of the range. We conclude that complete reproduction requires $m = 1$, *i.e.* that the two exposures be equal; but we must not forget that we have assumed the photographic procedure to be exactly the same, except as regards exposure.

Another reservation requires a moment's consideration. We have interpreted complete reproduction to demand equality of t'' and t. This seems to be in accord with usage; but it might be argued that *proportionality* of t'' and t' is all that is really required. For although the pictures considered in themselves differ, the effect upon the eye, or upon a photographic plate, may be made identical, all that is needed being a suitable variation in the intensity of the luminous background. But at this rate we should have to regard a white and a grey paper as equivalent.

If we abandon the restriction that the photographic process is to be the same in the two operations, simple conclusions of generality can hardly be looked for. But the problem is easily formulated. We may write

$$t' = f_1(et), \qquad t = t'' = f_2(e't'), \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where e, e' are the exposures, not generally equal, and f_1, f_2 represent two functions, whose forms may vary further with details of development and intensification. But for some printing processes f_2 might be treated as a fixed function. It would seem that this is the end at which discussion

should begin. When the printing process is laid down and the character of the results yielded thereby is determined, it becomes possible to say what is required in the negative; but it is not possible before.

In many photographs it would appear that gradation tends to be lost at the ends of the scale, that is in the high lights and deep shadows, and (as a necessary consequence, if the full range is preserved) to be exaggerated in the half-tones. For some purposes, where precise reproduction is not desired, this feature may be of advantage. Consider, for example, the experimental problem, discussed by Huggins, of photographing the solar corona without an eclipse. The corona is always present, but is overpowered by atmospheric glare. The problem is to render evident a very small relative difference of luminous intensity. If the difference is exaggerated in a suitably exposed and developed photograph, so much the better. A repetition of successive copyings might render conspicuous a difference originally invisible. At each operation we may suppose a factor a to be introduced, a being greater than unity. After n copyings dI/I becomes $a^n dI/I$. Unless the gain each time were very decided, this would be a slow process, and it would be liable to fail in practice owing to multiplication of slight irregular photographic markings. But a method proposed by Mach* and the present writer† should be of service here. By the aid of reflexion light at each stage is transmitted *twice* through the picture. By this means alone a is raised to equality with 2, and upon it any purely photographic exaggeration of gradation is superposed. Three successive copyings on this plan should ensure at least a ten-fold exaltation of contrast.

Another method, simpler in execution, consists in superposing a considerable number (n) of similar pictures. In this way the contrast is multiplied n times. Rays from a small, but powerful, source of light fall first upon a collimating lens, so as to traverse the pile of pictures as a parallel beam. Another condensing lens brings the rays to a focus, at which point the eye is placed. Some trials on this plan made a year ago gave promising results. Ten lantern-slides were prepared from a portrait negative. The exposure (to gas-light) was for about 3 seconds through the negative and for 30 seconds bare, *i.e.* with negative removed, and the development was rather light. On single plates the picture was but just visible. Some rough photometry indicated that each plate transmitted about one-third of the incident light. In carrying out the exposures suitable stops, cemented to the negative, must be provided to guide the lantern-plates into position, and thus to ensure their subsequent exact superposition by simple mechanical means.

When only a few plates are combined, the light of a Welsbach mantle suffices; but, as was to be expected, the utilization of the whole number (ten)

* Eder's *Jahrbuch f. Photographie.*

† *Phil. Mag.* Vol. XLIV. p. 282 (1897); *Scientific Papers*, Vol. IV. p. 333.

360.

ON THE PROPAGATION OF WAVES THROUGH A STRATIFIED MEDIUM, WITH SPECIAL REFERENCE TO THE QUESTION OF REFLECTION.

[*Proceedings of the Royal Society*, A, Vol. LXXXVI. pp. 207—266, 1912.]

THE medium is supposed to be such that its properties are everywhere a function of but one coordinate x, being of one uniform quality where x is less than a certain value x_1, and of another uniform quality (in general, different from the first) where x exceeds a greater value x_{m-1}; and the principal problem is the investigation of the reflection which in general ensues when plane waves in the first medium are incident upon the stratifications. For the present we suppose the quality to be uniform through strata of finite thickness, the first transition occurring when $x=x_1$, the second at $x=x_2$, and the last at $x=x_{m-1}$.

The expressions for the waves in the various media in order may be taken to be

$$\left.\begin{aligned}\phi_1 &= A_1 e^{i[ct+by-a_1(x-x_1)]} + B_1 e^{i[ct+by+a_1(x-x_1)]},\\ \phi_2 &= A_2 e^{i[ct+by-a_2(x-x_1)]} + B_2 e^{i[ct+by+a_2(x-x_1)]},\\ \phi_3 &= A_3 e^{i[ct+by-a_3(x-x_2)]} + B_3 e^{i[ct+by+a_3(x-x_2)]},\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(1)$$

and so on, the A's and B's denoting arbitrary constants. The first terms represent the waves travelling in the positive direction, the second those travelling in the negative direction; and our principal aim is the determination of the ratio B_1/A_1 imposed by the conditions of the problem, including the requirement that in the final medium there shall be no negative wave.

As in the simple transition from one uniform medium to another (*Theory of Sound*, § 270), the symbols c and b are common to all the media, the first depending merely upon the periodicity, while the constancy of the second is required in order that the traces of the various waves on the surfaces of

transition should move together—equivalent to the ordinary law of refraction. In the usual optical notation, if V be the velocity of propagation and θ the angle of incidence,

$$c = 2\pi V/\lambda, \qquad b = (2\pi/\lambda)\sin\theta, \qquad a = (2\pi/\lambda)\cos\theta, \quad \text{......(2)}$$

where V/λ, $\lambda^{-1}\sin\theta$ are the same in all the strata. On the other hand a is variable and is connected with the direction of propagation within the stratum by the relation

$$a = b\cot\theta. \quad \text{..................................(3)}$$

The a's are thus known in terms of the original angle of incidence and of the various refractive indices.

Since the factor $e^{i(ct+by)}$ runs through all our expressions, we may regard it as understood and write simply

$$\phi_1 = A_1 e^{-ia_1(x-x_1)} + B_1 e^{ia_1(x-x_1)}, \quad \text{........................(4)}$$

$$\phi_2 = A_2 e^{-ia_2(x-x_1)} + B_2 e^{ia_2(x-x_1)}, \quad \text{........................(5)}$$

$$\phi_3 = A_3 e^{-ia_3(x-x_2)} + B_3 e^{ia_3(x-x_2)}, \quad \text{........................(6)}$$

....

$$\phi_m = A_m e^{-ia_m(x-x_{m-1})} + B_m e^{ia_m(x-x_{m-1})}. \quad \text{..................(7)}$$

In the problem of reflection we are to make $B_m = 0$, and (if we please) $A_m = 1$.

We have now to consider the boundary conditions which hold at the surfaces of transition. In the case of sound travelling through gas, where ϕ is taken to represent the velocity-potential, these conditions are the continuity of $d\phi/dx$ and of $\sigma\phi$, where σ is the density. Whether the multiplier attaches to the dependent variable itself or to its derivative is of no particular significance. For example, if we take a new dependent variable ψ, equal to $\sigma\phi$, the above conditions are equivalent to the continuity of ψ and of $\sigma^{-1}d\psi/dx$. Nor should we really gain generality by introducing a multiplier in both places. We may therefore for the present confine ourselves to the acoustical form, knowing that the results will admit of interpretation in numerous other cases.

At the first transition $x = x_1$ the boundary conditions give

$$a_1(B_1 - A_1) = a_2(B_2 - A_2), \qquad \sigma_1(B_1 + A_1) = \sigma_2(B_2 + A_2). \quad \text{.......(8)}$$

If we stop here, we have the simple case of the juxtaposition of two media both of infinite depth. Supposing $B_2 = 0$, we get

$$\frac{B_1}{A_1} = \frac{\sigma_2/\sigma_1 - a_2/a_1}{\sigma_2/\sigma_1 + a_2/a_1} = \frac{\sigma_2/\sigma_1 - \cot\theta_2/\cot\theta_1}{\sigma_2/\sigma_1 + \cot\theta_2/\cot\theta_1}. \quad \text{..................(9)}$$

For a further discussion of (9) reference may be made to *Theory of Sound* (*loc. cit.*). In the case of the simple gases the compressibilities are

the same, and $\sigma_1 \sin^2\theta_1 = \sigma_2 \sin^2\theta_2$. The general formula (9) then identifies itself with Fresnel's expression

$$\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}. \quad \text{.............................(10)}$$

On the other hand, if $\sigma_2 = \sigma_1$, the change being one of compressibility only, we find

$$(9) = \frac{\sin(\theta_2-\theta_1)}{\sin(\theta_2+\theta_1)}, \quad \text{...........................(11)}$$

Fresnel's other expression.

In the above it is supposed that a_2 (and θ_2) are real. If the wave be incident in the more refractive medium and the angle of incidence be too great, a_2 becomes imaginary, say $-ia_2'$. In this case, of course, the reflection is total, the modulus of (9) becoming unity. The change of phase incurred is given by (9). In accordance with what has been said these results are at once available for the corresponding optical problems.

If there are more than two media, the boundary conditions at $x = x_2$ are

$$a_2\{B_2 e^{ia_2(x_2-x_1)} - A_2 e^{-ia_2(x_2-x_1)}\} = a_3(B_3 - A_3), \quad \text{...........(12)}$$

$$\sigma_2\{B_2 e^{ia_2(x_2-x_1)} + A_2 e^{-ia_2(x_2-x_1)}\} = \sigma_3(B_3 + A_3), \quad \text{...........(13)}$$

and so on. For extended calculations it is desirable to write these equations in an abbreviated shape. We set

$$B_2 - A_2 = H_2, \qquad B_2 + A_2 = K_2, \quad \text{etc., ...(14)}$$

$$\cos a_2(x_2 - x_1) = c_1, \qquad i\sin a_2(x_2 - x_1) = s_1, \quad \text{etc., ...(15)}$$

$$a_3/a_2 = \alpha_2, \qquad \sigma_3/\sigma_2 = \beta_2, \quad \text{etc.; ....(16)}$$

and the series of equations then takes the form

$$H_1 = \alpha_1 H_2, \qquad K_1 = \beta_1 K_2, \quad \text{...........(17)}$$

$$c_1 H_2 + s_1 K_2 = \alpha_2 H_3, \qquad s_1 H_2 + c_1 K_2 = \beta_2 K_3, \quad \text{...........(18)}$$

$$c_2 H_3 + s_2 K_3 = \alpha_3 H_4, \qquad s_2 H_3 + c_2 K_3 = \beta_3 K_4, \quad \text{...........(19)}$$

and so on. In the reflection problem the special condition is the numerical equality of H and K of highest suffix. We may make

$$H = -1, \qquad K = +1. \quad \text{.......................(20)}$$

As we have to work backwards from the terms of highest suffix, it is convenient to solve algebraically each pair of simple equations. In this way, remembering that $c^2 - s^2 = 1$, we get

$$H_1 = \alpha_1 H_2, \qquad K_1 = \beta_1 K_2, \quad \text{........(21)}$$

$$H_2 = c_1\alpha_2 H_3 - s_1\beta_2 K_3, \qquad K_2 = -s_1\alpha_2 H_3 + c_1\beta_2 K_3, \quad \text{........(22)}$$

$$H_3 = c_2\alpha_3 H_4 - s_2\beta_3 K_4, \qquad K_3 = -s_2\alpha_3 H_4 + c_2\beta_3 K_4, \quad \text{........(23)}$$

and so on. In these equations the c's and the β's are real, and also the α's, unless there is "total reflection"; the s's are pure imaginaries, with the same reservation.

When there are three media, we are to suppose in the problem of reflection that $H_3 = -1$, $K_3 = 1$. Thus from (21), (22),

$$H_1 = -\alpha_1 (c_1\alpha_2 + s_1\beta_2), \qquad K_1 = \beta_1 (s_1\alpha_2 + c_1\beta_2);$$

so that

$$\frac{B_1}{A_1} = \frac{K_1 + H_1}{K_1 - H_1} = \frac{c_1(\beta_1\beta_2 - \alpha_1\alpha_2) + s_1(\alpha_2\beta_1 - \alpha_1\beta_2)}{c_1(\beta_1\beta_2 + \alpha_1\alpha_2) + s_1(\alpha_2\beta_1 + \alpha_1\beta_2)}. \qquad \text{.........(24)}$$

If there be no "total reflection," the relative intensity of the reflected waves is

$$\frac{c_1^2(\beta_1\beta_2 - \alpha_1\alpha_2)^2 - s_1^2(\alpha_2\beta_1 - \alpha_1\beta_2)^2}{c_1^2(\beta_1\beta_2 + \alpha_1\alpha_2)^2 - s_1^2(\alpha_2\beta_1 + \alpha_1\beta_2)^2}, \qquad \text{.................(25)}$$

where

$$c_1^2 = \cos^2 a_2(x_2 - x_1), \qquad -s_1^2 = \sin^2 a_2(x_2 - x_1). \qquad \text{.........(26)}$$

The reflection will vanish independently of the values of c_1 and s_1, *i.e.*, whatever may be the thickness of the middle layer, provided

$$\beta_1\beta_2 - \alpha_1\alpha_2 = 0, \quad \alpha_2\beta_1 - \alpha_1\beta_2 = 0; \quad \text{or} \quad \beta_1 = \alpha_1, \quad \beta_2 = \alpha_2,$$

since these quantities are all positive. Reference to (9) shows that these are the conditions of vanishing reflection at the two surfaces of transition considered separately.

If these conditions be not satisfied, the evanescence of (25) requires that either c_1 or s_1 be zero. The latter case is realized if the intermediate layer be abolished, and the remaining condition is equivalent to $\sigma_3/\sigma_1 = a_3/a_1$, as was to be expected from (9). We learn now that, if there would be no reflection in the absence of an intermediate layer, its introduction will have no effect provided $a_2(x_2 - x_1)$ be a multiple of π. An obvious example is when the first and third media are similar, as in the usual theory of "thin plates."

On the other hand, if c_1, or $\cos a_2(x_2 - x_1)$, vanish, the remaining requirement for the evanescence of (25) is that $\beta_2/\alpha_2 = \beta_1/\alpha_1$.

In this case

$$\frac{\beta_1 - \alpha_1}{\beta_1 + \alpha_1} = \frac{\beta_2 - \alpha_2}{\beta_2 + \alpha_2};$$

so that by (9) the reflections at the two faces are equal in all respects.

In general, if the third and first media are similar, (25) reduces to

$$\frac{\{\beta_1/\alpha_1 - \alpha_1/\beta_1\}^2 \sin^2 a_2(x_2 - x_1)}{4\cos^2 a_2(x_2 - x_1) + \{\beta_1/\alpha_1 + \alpha_1/\beta_1\}^2 \sin^2 a_2(x_2 - x_1)}, \qquad \text{.........(27)}$$

which may readily be identified with the expression usually given in terms of (9).

It remains to consider the cases of so-called total reflection. If this occurs only at the *second* surface of transition, a_1, a_2 are real, while a_3 is a

pure imaginary. Thus α_1 is real, and α_2 is imaginary; c_1 is real always, and s_1 is imaginary as before; the β's are always real. Thus, if we separate the real and imaginary parts of the numerator and denominator of (24), we get

$$\frac{B_1}{A_1}=\frac{c_1\beta_1\beta_2+s_1\alpha_2\beta_1-c_1\alpha_1\alpha_2-s_1\alpha_1\beta_2}{c_1\beta_1\beta_2+s_1\alpha_2\beta_1+c_1\alpha_1\alpha_2+s_1\alpha_1\beta_2}, \quad \text{..............(28)}$$

of which the modulus is unity. In this case, accordingly, the reflection back in the first medium is literally total, whatever may be the thickness of the intermediate layer, as was to be expected.

The separation of real and imaginary parts follows the same rule when a_2 is imaginary, as well as a_3. For then α_1 is imaginary, while α_2, s_1 are real. Thus $s_1\alpha_2\beta_1$ remains real, and $c_1\alpha_1\alpha_2$, $s_1\alpha_1\beta_2$ remain imaginary. The reflection back in the first medium is total in this case also.

The only other case requiring consideration occurs when a_2 is imaginary and a_3 real. The reflection is then total if the middle layer be thick enough, but if this thickness be reduced, the reflection cannot remain total, as is evident if we suppose the thickness to vanish. The ratios α_1, α_2 are now both imaginary, while s_1 is real. The separation of real and imaginary parts stands as in (24), and the intensity of reflection is still expressed by (25). If we take $a_2=-ia_2'$, we may write in place of (25),

$$\frac{(\beta_1\beta_2-\alpha_1\alpha_2)^2\cosh^2 a_2'(x_2-x_1)-(\alpha_2\beta_1-\alpha_1\beta_2)^2\sinh^2 a_2'(x_2-x_1)}{(\beta_1\beta_2+\alpha_1\alpha_2)^2\cosh^2 a_2'(x_2-x_1)-(\alpha_2\beta_1+\alpha_1\beta_2)^2\sinh^2 a_2'(x_2-x_1)}. \quad \text{...(29)}$$

When x_2-x_1 is extremely small, this reduces to

$$\frac{(\beta_1\beta_2-\alpha_1\alpha_2)^2}{(\beta_1\beta_2+\alpha_1\alpha_2)^2}, \quad \text{or} \quad \frac{(\sigma_3/\sigma_1-a_3/a_1)^2}{(\sigma_3/\sigma_1+a_3/a_1)^2},$$

in accordance with (9).

When on the other hand x_2-x_1 exceeds a few wave-lengths, (29) approaches unity, as we see from a form, equivalent to (29), viz.,

$$\frac{(\beta_1^2-\alpha_1^2)(\beta_2^2-\alpha_2^2)\cosh^2 a_2'(x_2-x_1)+(\alpha_2\beta_1-\alpha_1\beta_2)^2}{(\beta_1^2-\alpha_1^2)(\beta_2^2-\alpha_2^2)\cosh^2 a_2'(x_2-x_1)+(\alpha_2\beta_1+\alpha_1\beta_2)^2}. \quad \text{......(30)}$$

It is to be remembered that in (30), α_1^2, α_2^2, $\alpha_1\alpha_2$ have negative values.

The form assumed when the third medium is similar to the first may be noted. In this case $\alpha_1\alpha_2=1$, $\beta_1\beta_2=1$, and we get from (29)

$$\frac{(\beta_1/\alpha_1-\alpha_1/\beta_1)^2\sinh^2 a_2'(x_2-x_1)}{(\beta_1/\alpha_1-\alpha_1/\beta_1)^2\sinh^2 a_2'(x_2-x_1)-4}. \quad \text{..................(31)}$$

In this case, of course, the reflection vanishes when x_2-x_1 is sufficiently reduced.

Equations (21), etc., may be regarded as constituting the solution of the general problem. If there are m media, we suppose $H_m=-1$, $K_m=1$,

requires a more powerful source. Good results were obtained with a lime-light; the portrait, barely visible at all on the single plates, came out fairly well under this illumination. If it were proposed to push the experiment much further by the combination of a larger number of plates, it would probably be advantageous to immerse them in benzole contained in a tank, so as to obviate the numerous reflexions at the surfaces.

It has been mentioned that in the above experiment the development of the plates was rather light. The question may be raised whether further development, or intensification, might not make one plate as good as two or three superposed. I think that to a certain extent this is so. When in a recent experiment one of the plates above described was intensified with mercuric chloride followed by ferrous oxalate, the picture was certainly more apparent than before, when backed by a sufficiently strong light. And the process of intensification may be repeated. But there is another point to be considered. In the illustrative experiment it was convenient to copy all the plates from the same negative. But this procedure would not be the proper one in an attempt to render visible the solar corona. For this purpose a good many independent pictures should be combined, so as to eliminate slight photographic defects. As in many physical measurements, when it is desired to enhance the delicacy, the aim must be to separate feeble constant effects from chance disturbances.

It may be that, besides that of the corona, there are other astronomical problems to which one or other of the methods above described, or a combination of both, might be applied with a prospect of attaining a further advance.

thence calculate in order from the pairs of simple equations H_{m-1}, K_{m-1}; H_{m-2}, K_{m-2}, etc., until H_1 and K_1 are reached; and then determine the ratio B_1/A_1. The procedure would entail no difficulty in any special case numerically given; but the algebraic expression of H_1 and K_1 in terms of H_m and K_m soon becomes complicated, unless further simplifying conditions are introduced. Such simplification may be of two kinds. In the first it is supposed that the total thickness between the initial and final media is small relatively to the wave-lengths, so that the phase-changes occurring within the layer are of subordinate importance. In the second kind of simplification the thicknesses are left arbitrary, but the changes in the character of the medium, which occur at each transition, are supposed small.

The problem of a thin transitional layer has been treated by several authors, L. Lorenz*, Van Ryn†, Drude‡, Schott§, and Maclaurin‖. A full account will be found in *Theory of Light* by the last named. It will therefore not be necessary to treat the subject in detail here; but it may be worth while to indicate how the results may be derived from our equations. For this purpose it is convenient to revert to the original notation so far as to retain a and σ. Thus in place of (17), etc., we write

$$a_1 H_1 = a_2 H_2, \qquad \sigma_1 K_1 = \sigma_2 K_2, \quad \text{.........(32)}$$

$$a_2(c_1 H_2 + s_1 K_2) = a_3 H_3, \qquad \sigma_2(s_1 H_2 + c_1 K_2) = \sigma_3 K_3, \text{ etc. } \quad \text{...(33)}$$

In virtue of the supposition that all the layers are thin, the c's are nearly equal to unity and the s's are small. Thus, for a first approximation, we identify c with 1 and neglect s altogether, so obtaining

$$a_1 H_1 = a_2 H_2 = \ldots = a_m H_m, \qquad \sigma_1 K_1 = \sigma_2 K_2 = \ldots = \sigma_m K_m. \quad \text{...(34)}$$

The relation of H_1, K_1 to H_m, K_m is the same as if the transition between the extreme values took place without intermediate layers, and the law of reflection is not disturbed by the presence of these layers, as was to be expected.

For the second approximation we may still identify the c's with unity, while the s's are retained as quantities of the first order. Adding together the column of equations constituting the first members of (32), (33), etc., we find

$$a_1 H_1 + a_2 s_1 K_2 + a_3 s_2 K_3 + \ldots + a_{m-1} s_{m-2} K_{m-1} = a_m H_m; \quad \text{......(35)}$$

and in like manner, with substitution of σ for a and interchange of K and H,

$$\sigma_1 K_1 + \sigma_2 s_1 H_2 + \ldots + \sigma_{m-1} s_{m-2} H_{m-1} = \sigma_m K_m. \quad \text{.........(36)}$$

* *Pogg. Ann.* 1860, Vol. CXI. p. 460.
† *Wied. Ann.* 1883, Vol. XX. p. 22.
‡ *Wied. Ann.* 1891, Vol. XLIII. p. 126.
§ *Phil. Trans.* 1894, Vol. CLXXXV. p. 823.
‖ *Roy. Soc. Proc.* A, 1905, Vol. LXXVI. p. 49.

In the small terms containing s's we may substitute the approximate values of H and K from (34). For the problem of reflection we suppose $H_m + K_m = 0$. Hence

$$\frac{H_1}{K_1} = -\frac{\sigma_1 a_m}{\sigma_m a_1} \frac{1 + \dfrac{\sigma_m}{a_m} \Sigma \dfrac{a_2 s_1}{\sigma_2}}{1 + \dfrac{a_m}{\sigma_m} \Sigma \dfrac{\sigma_2 s_1}{a_2}}. \quad \text{.....................(37)}$$

In (37), $s_1 = ia_2(x_2 - x_1)$, and so on, so that

$$\Sigma \frac{a_2 s_1}{\sigma_1} = i \int \frac{a^2 dx}{\sigma}, \qquad \Sigma \frac{\sigma_2 s_1}{a_2} = i \int \sigma\, dx, \quad \text{................(38)}$$

the integration extending over the layer of transition.

One conclusion may be drawn at once. To this degree of approximation the reflection is independent of the order of the strata. It will be noted that the sums in (37) are pure imaginaries. In what follows we shall suppose that a_m is real.

As the final result for the reflection, we find

$$\frac{B_1}{A_1} = \frac{K_1 + H_1}{K_1 - H_1} = Re^{i\delta}, \quad \text{..............................(39)}$$

where
$$R = \frac{\sigma_m/\sigma_1 - a_m/a_1}{\sigma_m/\sigma_1 + a_m/a_1}, \quad \text{..............................(40)}$$

$$\tan\delta = 2\frac{\dfrac{a_m}{\sigma_m}\displaystyle\int \sigma\, dx - \dfrac{\sigma_m}{a_m}\displaystyle\int \frac{a^2 dx}{\sigma}}{\dfrac{a_1 \sigma_m}{a_m \sigma_1} - \dfrac{a_m \sigma_1}{a_1 \sigma_m}}. \quad \text{...................(41)}$$

To this order of approximation the *intensity* of the reflection is unchanged by the presence of the intermediate layers, unless, indeed, the circumstances are such that (40) is itself small. If $\sigma_m/\sigma_1 = a_m/a_1$ absolutely, we have

$$R = \tfrac{1}{2}\left\{\frac{a_m}{\sigma_m}\int \sigma\, dx - \frac{\sigma_m}{a_m}\int \frac{a^2 dx}{\sigma}\right\} \quad \text{...................(42)}$$

and $\delta = \frac{1}{2}\pi$. This case is important in Optics, as representing the reflection at the polarising angle from a contaminated surface.

The two important optical cases: (i) where σ is constant, leading (when there is no transitional layer) to Fresnel's formula (11), and (ii) where $\sigma \sin^2\theta$ is constant, leading to (10), are now easily treated as special examples. Introducing the refractive index μ, we find after reduction for case (i)

$$\delta' = -\frac{4\pi \cos\theta_1}{\lambda_1(\mu_m^2 - \mu_1^2)}\int (\mu_m^2 - \mu^2)\, dx, \quad \text{................(43)}$$

where λ_1, μ_1 relate to the first medium, μ_m is the index for the last medium, and the integration is over the layer of transition. The application of (43)

should be noticed when the layer is in effect abolished, either by supposing $\mu = \mu_m$, or, on the other hand, $\mu = \mu_1$.

In the second case (42), corresponding to the polarising angle, becomes

$$R = \frac{\pi}{\lambda_1 \mu_1 \sqrt{(\mu_m^2 + \mu_1^2)}} \int \frac{(\mu^2 - \mu_1^2)(\mu^2 - \mu_m^2)}{\mu^2} dx. \quad \text{.........(44)}$$

In general for this case

$$\delta'' = -\frac{4\pi \cos\theta_1}{\lambda_1} \frac{\cos^2\theta_1 \int (\mu_m^2 - \mu^2)\, dx - \sin^2\theta_1 \int (\mu_m^2 - \mu^2)\left(\frac{\mu_1^2}{\mu^2} + \frac{\mu_1^2}{\mu_m^2} - 1\right) dx}{(\mu_m^2 - \mu_1^2)\left(\cos^2\theta_1 - \frac{\mu_1^2}{\mu_m^2}\sin^2\theta_1\right)}. \quad \text{......(45)}$$

The second fraction in (45) is equal to the thickness of the layer of transition simply, when we suppose $\mu = \mu_1$.

Further,
$$\delta'' - \delta' = -\frac{4\pi}{\lambda_1} \frac{\cos\theta_1 \sin^2\theta_1}{\mu_m^2 - \mu_1^2} \frac{\int \frac{(\mu_m^2 - \mu^2)(\mu^2 - \mu_1^2)}{\mu} dx}{\cos^2\theta_1 - \frac{\mu_1^2}{\mu_m^2}\sin^2\theta_1}, \quad \text{......(46)}$$

the difference of phase vanishing, as it ought to do, when $\mu = \mu_1$, or μ_m, or again, when $\theta_1 = 0$.

It should not escape notice that the expressions (10) and (11) have different signs when θ_1 and θ_2 are small. This anomaly, as it must appear from an optical point of view, should be corrected when we consider the significance of $\delta'' - \delta'$. The origin of it lies in the circumstance that, in our application of the boundary conditions, we have, in effect, used different vectors as dependent variables to express light of the two polarisations. For further explanation reference may be made to former writings, *e.g.* "On the Dynamical Theory of Gratings*."

If throughout the range of integration, μ is intermediate between the terminal values μ_1, μ_m, the reflection is of the kind called positive by Jamin. The transition may well be of this character when there is no contamination. On the other hand, the reflection is negative, if μ has throughout a value outside the range between μ_1 and μ_m. It is probable that something of this kind occurs when water has a greasy surface.

The formulæ required in Optics, viz. (43), (44), (45), (46), are due, in substance, to Lorenz and Van Ryn. The more general expressions (41), (42) do not seem to have been given.

There is no particular difficulty in pursuing the approximation from (32), etc. At the next stage the second term in the expansion of the c's

* *Roy. Soc. Proc.* A, 1907, Vol. LXXIX. p. 413.

must be retained, while the s's are still sufficiently represented by the first terms. The result, analogous to (37), (38), is

$$\frac{H_1}{K_1} = -\frac{\sigma_1 a_m}{\sigma_m a_1} \frac{1 - \int_0^d \sigma \,.\int_0^x \frac{a^2 dx}{\sigma} \,.\, dx + i\frac{\sigma_m}{a_m}\int_0^d \frac{a^2}{\sigma} dx}{1 - \int_0^d \frac{a^2}{\sigma}\,.\int_0^x \sigma dx \,.\, dx + i\frac{a_m}{\sigma_m}\int_0^d \sigma dx}, \quad \text{.........(47)}$$

in which the terminal abscissæ of the variable layer are taken to be 0 and d, instead of x_1 and x_{m-1}. I do not follow out the application to particular cases such as σ = constant, or $\sigma \sin^2\theta$ = constant. For this reference may be made to Maclaurin, who, however, uses a different method.

The second case which allows of a simple approximate expression for the reflection arises when all the partial reflections are small. It is then hardly necessary to appeal to the general equations: the method usually employed in Optics suffices. The assumptions are that at each surface of transition the incident waves may be taken to be the same as in the first medium, merely retarded by the appropriate amount, and that each partial reflection reaches the first medium no otherwise modified than by such retardation. This amounts to the neglect of waves three times reflected. Thus

$$\frac{B_1}{A_1} = \frac{\beta_1 - \alpha_1}{\beta_1 + \alpha_1} + \frac{\beta_2 - \alpha_2}{\beta_2 + \alpha_2} e^{-2i\alpha_2(x_2 - x_1)} + \frac{\beta_3 - \alpha_3}{\beta_3 + \alpha_3} e^{-2i\{\alpha_2(x_2 - x_1) + \alpha_3(x_3 - x_2)\}} + \dots \quad (48)$$

An interesting question suggests itself as to the manner in which the transition from one uniform medium to another must be effected in order to obviate reflection, and especially as to the least thickness of the layer of transition consistent with this result. If there be two transitions only, the least thickness of the layer is obtained by supposing in (48)

$$\frac{\beta_1 - \alpha_1}{\beta_1 + \alpha_1} = \frac{\beta_2 - \alpha_2}{\beta_2 + \alpha_2} \quad \text{.............................(49)}$$

and

$$2\alpha_2(x_2 - x_1) = \pi; \quad \text{.............................(50)}$$

and this conclusion, as we have seen already, is not limited to the case of small differences of quality. In its application to perpendicular incidence, (50) expresses that the thickness of the layer is one-quarter of the wave-length proper to the layer. The two partial reflections are equal in magnitude and sign. It is evident that nothing better than this can be done so long as the reflections are of one sign, however numerous the surfaces of transition may be.

If we allow the partial reflections to be of different signs, some reduction of the necessary thickness is possible. For example, suppose that there are two intermediate layers of equal thickness, of which the first is similar to the final uniform medium, and the second similar to the initial uniform medium. Of the three partial reflections the first and third are similar, but the second

is of the opposite sign. If three vectors of equal numerical value compensate one another, they must be at angles of 120°. The necessary conditions are satisfied (in the case of perpendicular transmission) if the total thickness ($2l$) is $\frac{1}{6}\lambda$, in accordance with

$$1 - e^{-4\pi i l/\lambda} + e^{-8\pi i l/\lambda} = 0.$$

The total thickness of the layer of transition is thus somewhat reduced, but only by a very artificial arrangement, such as would not usually be contemplated when a layer of transition is spoken of. If the progress from the first to the second uniform quality be always *in one direction*, reflection cannot be obviated unless the layer be at least $\frac{1}{4}\lambda$ thick.

The general formula (48) may be adapted to express the result appropriate to continuous variation of the medium. Suppose, for example, that σ is constant, making $\beta = 1$, and corresponding to the continuity of both ϕ and $d\phi/dx$*. It is convenient to suppose that the variation commences at $x = 0$. Then (48) may be written

$$\frac{B_1}{A_1} = -\int \frac{da}{2a} e^{-2i\int_0^x a\,dx}, \quad \text{.......................(51)}$$

a at any point x being connected with the angle of propagation by the usual relation (3). In the special case of perpendicular propagation, $a = 2\pi\mu/\lambda_1\mu_1$, μ being refractive index and λ_1, μ_1 relating to the first medium.

A curious example, theoretically possible even if unrealizable in experiment, arises when the variable medium is constituted in such a manner that the velocity of propagation is everywhere constant, so that there is no refraction. Then a is constant, $\alpha = 1$, and (48) gives

$$\frac{B_1}{A_1} = \int \frac{d\sigma}{2\sigma} e^{-2iax}. \quad \text{............................(52)}$$

Some of the questions relating to the propagation of waves in a variable medium are more readily treated on the basis of the appropriate differential equation. As in (1), we suppose that the waves are plane, and that the medium is stratified in plane strata perpendicular to x, and we usually omit the exponential factors involving t and y, which may be supposed to run through. In the case of perpendicular propagation, y would not appear at all.

Consider the differential equation

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0, \quad \text{............................(53)}$$

in which (unless k^2 can be infinite) it is necessary to suppose that ϕ and $d\phi/dx$ are continuous; k^2 is a function of x, which must be everywhere

* These would be the conditions appropriate to a stretched string of variable longitudinal density vibrating transversely.

positive when the transmission is perpendicular, as, for example, in the case of a stretched string. When the transmission is oblique to the strata, k^2 may become negative, corresponding to "total reflection," but in most of what follows we shall assume that this does not happen. The continuity of ϕ and $d\phi/dx$, even though k^2 be discontinuous, appears to limit the application of (53) to certain kinds of waves, although, as a matter of analysis, the general differential equation of the second order may always be reduced to this form*.

In the theory of a uniform medium, we may consider stationary waves or progressive waves. The former may be either

$$\phi = A \cos k_0 x \cos pt, \quad \text{or} \quad \phi = B \sin k_0 x \sin pt;$$

and, if $B = \pm A$, the two may be combined, so as to constitute progressive waves

$$\phi = A \cos (pt \pm k_0 x).$$

Conversely, progressive waves, travelling in opposite directions, may be combined so as to constitute stationary waves. When we pass to variable media, no ambiguity arises respecting stationary waves; they are such that the phase is the same at all points. But is there such a thing as a progressive wave? In the full sense of the phrase there is not. In general, if we contemplate the wave forms at two different times, the difference between them cannot be represented by a mere shift of position proportional to the interval of time which has elapsed.

The solution of (53) may be taken to be

$$\phi = A'\psi(x) + B'\chi(x), \quad \text{......................(54)}$$

where $\psi(x)$, $\chi(x)$ are real oscillatory functions of x; A', B', arbitrary constants as regards x. If we introduce the time-factor, writing p in place of the less familiar c of (1), we may take

$$\phi = A \cos pt \,.\, \psi(x) + B \sin pt \,.\, \chi(x); \quad \text{...............(55)}$$

and this may be put into the form

$$\phi = H \cos(pt - \theta), \quad \text{..........................(56)}$$

where $$H \cos\theta = A\psi(x), \quad H \sin\theta = B\chi(x), \quad \text{...............(57)}$$

or $$H^2 = A^2[\psi(x)]^2 + B^2[\chi(x)]^2, \quad \text{...................(58)}$$

$$\theta = \tan^{-1}\frac{B\,.\,\chi(x)}{A\,.\,\psi(x)}. \quad \text{..........................(59)}$$

But the expression for ϕ in (56) cannot be said to represent in general a progressive wave. We may illustrate this even from the case of the uniform medium where $\psi(x) = \cos kx$, $\chi(x) = \sin kx$. In this case (56) becomes

$$\phi = \{A^2 \cos^2 kx + B^2 \sin^2 kx\}^{\frac{1}{2}} \cos\left\{pt - \tan^{-1}\left(\frac{B}{A}\tan kx\right)\right\}. \quad \text{...(60)}$$

* Forsyth's *Differential Equations*, § 59.

If $B = \pm A$, reduction ensues to the familiar positive or negative progressive wave. But if B be not equal to $\pm A$, (55), taking the form

$$\phi = \tfrac{1}{2}(A + B)\cos(pt - kx) + \tfrac{1}{2}(A - B)\cos(pt + kx),$$

clearly does not represent a progressive wave. The mere possibility of reduction to the form (57) proves little, without an examination of the character of H and θ.

It may be of interest to consider for a moment the character of θ in (60). If B/A, or, say, m, is positive, θ may be identified with kx at the quadrants but elsewhere they differ, unless $m = 1$. Introducing the imaginary expressions for tangents, we find

$$\theta = kx + M\sin 2kx + \tfrac{1}{2}M^2 \sin 4kx + \tfrac{1}{3}M^3 \sin 6kx + \ldots, \quad \text{......(61)}$$

where

$$M = \frac{m-1}{m+1}. \quad \text{......(62)}$$

When k is constant, one of the solutions of (53) makes ϕ proportional to e^{-ikx}. Acting on this suggestion, and following out optical ideas, let us assume in general

$$\phi = \eta e^{-i\int a\,dx}, \quad \text{......(63)}$$

where the amplitude η and a are real functions of x, which, for the purpose of approximations, may be supposed to vary slowly. Substituting in (53), we find

$$\frac{d^2\eta}{dx^2} + (k^2 - a^2)\eta - 2ia^{\frac{1}{2}}\frac{d}{dx}(a^{\frac{1}{2}}\eta) = 0. \quad \text{......(64)}$$

For a first approximation, we neglect $d^2\eta/dx^2$. Hence

$$k = a, \qquad k^{\frac{1}{2}}\eta = C, \quad \text{......(65)}$$

so that

$$\phi = Ck^{-\frac{1}{2}} e^{ipt} e^{-i\int k\,dx}, \quad \text{......(66)}$$

or in real form,

$$\phi = Ck^{-\frac{1}{2}}\cos(pt - \textstyle\int k\,dx). \quad \text{......(67)}$$

If we hold rigorously to the suppositions expressed in (65), the satisfaction of (64) requires that $d^2\eta/dx^2 = 0$, or $d^2k^{-\frac{1}{2}}/dx^2 = 0$. With omission of arbitrary constants affecting merely the origin and the scale of x, this makes $k^2 = x^{-4}$, corresponding to the differential equation

$$x^4\frac{d^2\phi}{dx^2} + \phi = 0, \quad \text{......(68)}$$

whose accurate solution is accordingly

$$\phi = Cxe^{i(pt - 1/x)}. \quad \text{......(69)}$$

In (69) the imaginary part may be rejected. The solution (69) is, of course, easily verified. In all other cases (67) is only an approximation.

As an example, the case where $k^2 = n^2/x^2$ may be referred to. Here $\int k dx = n \log x - \epsilon$, and (67) gives

$$\phi = Cx^{\frac{1}{2}} \cos(pt - n \log x + \epsilon) \qquad (70)$$

as an approximate solution. We shall see presently that a slight change makes it accurate.

Reverting to (64), we recognize that the first and second terms are real, while the third is imaginary. The satisfaction of the equation requires therefore that

$$a^{\frac{1}{2}}\eta = C, \qquad (71)$$

and that

$$k^2 = C^4\eta^{-4} - \frac{1}{\eta}\frac{d^2\eta}{dx^2}; \qquad (72)$$

while (63) becomes

$$\phi = \eta e^{-iC^2\int \eta^{-2} dx}. \qquad (73)$$

Let us examine in what cases η may take the form Dx^r. From (72),

$$k^2 = C^4 D^{-4} x^{-4r} - r(r-1)x^{-2}. \qquad (74)$$

If $r = 0$, k^2 is constant. If $r = 1$, $k^2 = C^4 D^{-4} x^{-4}$, already considered in (68). The only other case in which k^2 is a simple power of x occurs when $r = \frac{1}{2}$, making

$$k^2 = (C^4 D^{-4} + \tfrac{1}{4})x^{-2} = n^2 x^{-2} \text{ (say)}. \qquad (75)$$

Here $\eta = Dx^{\frac{1}{2}}$, $C^2\int \eta^{-2}\, dx = C^2/D^2 . \log x - \epsilon$, and the realized form of (73) is

$$\phi = Dx^{\frac{1}{2}} \cos\{pt - \sqrt{(n^2 - \tfrac{1}{4})} \log x + \epsilon\}, \qquad (76)$$

which is the exact form of the solution obtained by approximate methods in (70). For a discussion of (76) reference may be made to *Theory of Sound*, second edition, § 148 *b*.

The relation between a and η in (71) is the expression of the energy condition, as appears readily if we consider the application to waves along a stretched string. From (53), with restoration of e^{ipt},

$$\frac{d\phi}{dt} = e^{ipt} e^{-i\int a dx} . ip\eta, \qquad \frac{d\phi}{dx} = e^{ipt} e^{-i\int a dx}\left\{\frac{d\eta}{dx} - ia\eta\right\}.$$

If the common phase factors be omitted, the parts of $d\phi/dt$ and $d\phi/dx$ which are in the same phase are as $p\eta$ and $a\eta$, and thus the mean work transmitted at any place is as $a\eta^2$. Since there is no accumulation of energy between two places, $a\eta^2$ must be constant.

When the changes are gradual enough, a may be identified with k, and then $\eta \propto k^{-\frac{1}{2}}$, as represented in (67).

If we regard η as a given function of x, a follows when C has been chosen, and also k^2 from (72). In the case of perpendicular propagation k^2 cannot be negative, but this is the only restriction. When η is constant, k^2 is constant;

and thus if we suppose η to pass from one constant value to another through a finite transitional layer, the transition is also from one uniform k^2 to another; and (73) shows that there is no reflection back into the first medium. If the terminal values of η and therefore of k^2 be given, and the transitional layer be thick enough, it will always be possible, and that in an infinite number of ways, to avoid a negative k^2, and thus to secure complete transmission without reflection back; but if with given terminal values the layer be too much reduced, k^2 must become negative. In this case reflection cannot be obviated.

It may appear at first sight as if this argument proved too much, and that there should be no reflection in any case so long as k^2 is positive throughout. But although a constant η requires a constant k^2, it does not follow conversely that a constant k^2 requires a constant η, and, in fact, this is not true. *One* solution of (72), when k^2 is constant, certainly is $\eta^2 = C^2/k$; but the complete solution necessarily includes two arbitrary constants, of which C is not one. From (60) it may be anticipated that a solution of (72) may be

$$\eta^2 = A^2 \cos^2 kx + B^2 \sin^2 kx = \tfrac{1}{2}(A^2 + B^2) + \tfrac{1}{2}(A^2 - B^2)\cos 2kx. \quad \ldots (77)$$

From this we find on differentiation

$$\eta^3 \frac{d^2\eta}{dx^2} + k^2\eta^4 = k^2 A^2 B^2;$$

and thus (72) is satisfied, provided that

$$k^2 A^2 B^2 = C^2. \quad \ldots\ldots (78)$$

It appears then that (77) subject to (78) is a solution of (72). The second arbitrary constant evidently takes the form of an arbitrary addition to x, and η will not be constant unless $A^2 = B^2$.

On the supposition that η and a are slowly varying functions, the approximations of (65) may be pursued. We find

$$\eta = Ck^{-\frac{1}{2}}\left\{1 - \tfrac{1}{4}k^{-\frac{3}{2}}\frac{d^2k^{-\frac{1}{2}}}{dx^2}\right\}, \quad \ldots\ldots (79)$$

$$a = k + \tfrac{1}{2}k^{-\frac{1}{2}}\frac{d^2k^{-\frac{1}{2}}}{dx^2}. \quad \ldots\ldots (80)$$

The retardation, as usually reckoned in optics, is $\int k\,dx$. The additional retardation according to (80) is

$$\tfrac{1}{2}\int k^{-\frac{1}{2}}\frac{d^2k^{-\frac{1}{2}}}{dx^2}\,dx = \tfrac{1}{2}\left[k^{-\frac{1}{2}}\frac{dk^{-\frac{1}{2}}}{dx}\right] - \tfrac{1}{2}\int\left(\frac{dk^{-\frac{1}{2}}}{dx}\right)^2 dx.$$

As applied to the transition from one uniform medium to another, the retardation is *less* than according to the first approximation by

$$\tfrac{1}{2}\int\left(\frac{dk^{-\frac{1}{2}}}{dx}\right)^2 dx. \quad \ldots\ldots (81)$$

The supposition that η varies slowly excludes more than a very small reflection.

Equations (79), (80) may be tested on the particular case already referred to where $k = n/x$. We get

$$\eta = Cn^{-\frac{1}{2}} x^{\frac{1}{2}} \left(1 + \frac{1}{16n^2}\right), \qquad a = \frac{1}{x}\left(n - \frac{1}{8n}\right);$$

so that
$$\int a\,dx = \left(n - \frac{1}{8n}\right) \log x.$$

When n^{-4} is neglected in comparison with unity, $n - \frac{1}{8}n^{-1}$ may be identified with $\sqrt{(n^2 - \frac{1}{4})}$.

Let us now consider what are the possibilities of avoiding reflection when the transition layer $(x_2 - x_1)$ between two uniform media is reduced. If η_1, k_1; η_2, k_2 are the terminal values, (79) requires that

$$k_1^2 = C^4 \eta_1^{-4}, \qquad k_2^2 = C^4 \eta_2^{-4}.$$

We will suppose that $\eta_2 > \eta_1$. If the transition from η_1 to η_2 be made too quickly, viz., in too short a space, $d^2\eta/dx^2$ will become somewhere so large as to render k^2 negative. The same consideration shows that at the beginning of the layer of transition (x_1), $d\eta/dx$ must vanish. The quickest admissible rise of η will ensue when the curve of rise is such as to make k^2 vanish. When η attains the second prescribed value η_2, it must suddenly become constant, notwithstanding that this makes k^2 positively infinite.

From (72) it appears that the curve of rise thus defined satisfies

$$\frac{d^2\eta}{dx^2} = C^4 \eta^{-3}. \qquad \text{.................................(82)}$$

The solution of (82), subject to the conditions that $\eta = \eta_1$, $d\eta/dx = 0$, when $x = x_1$, is

$$\eta^2 - \eta_1^2 = C^4 \eta_1^{-2} (x - x_1)^2 = k_1^2 \eta_1^2 (x - x_1)^2. \qquad \text{..............(83)}$$

Again, when $\eta = \eta_2$, $x = x_2$, so that

$$k_1^2 (x_2 - x_1)^2 = \frac{\eta_2^2 - \eta_1^2}{\eta_1^2} = \frac{k_1 - k_2}{k_2}, \qquad \text{....................(84)}$$

giving the minimum thickness of the layer of transition.

It will be observed that the minimum thickness of the layer of transition necessary to avoid reflection diminishes without limit with $k_1 - k_2$, that is, as the difference between the two media diminishes. However, the arrangement under discussion is very artificial. In the case of the string, for example, it is supposed that the density drops suddenly from the first uniform value to zero, at which it remains constant for a time. At the end of this it becomes momentarily infinite, before assuming the second uniform value. The infinite longitudinal density at x_2 is equivalent to a finite load

there attached. In the layer of transition, if so it may be called, the string remains straight during the passage of the waves.

If, as in the more ordinary use of the term, we require the transition to be such that k^2 moves always in one direction from the first terminal value to the second, the problem is one already considered. The minimum thickness is such that k^2 has throughout it a constant intermediate value, so chosen as to make the reflections equal at the two faces.

It would be of interest to consider a particular case in which k^2 varies continuously and always in the one direction. As appears at once from (72), $d^2\eta/dx^2$, as well as $d\eta/dx$, must vanish at both ends of the layer, and there must also be a third point of inflection between. If the layer be from $x=0$ to $x=\beta$, we may take

$$\frac{d^2\eta}{dx^2} = Ax(x-\alpha)(x-\beta). \quad \text{.......................(85)}$$

We find that $\beta = 2\alpha$, and that

$$\frac{\eta-\eta_1}{\eta_2-\eta_1} = \frac{15x^3}{4\alpha^5}\left\{\frac{x^2}{20} - \frac{\alpha x}{4} + \frac{\alpha^2}{3}\right\}, \quad \text{.....................(86)}$$

$$\frac{1}{\eta_2-\eta_1}\frac{d^2\eta}{dx^2} = \frac{15x}{4\alpha^5}(x-\alpha)(x-2\alpha). \quad \text{..................(87)}$$

From these k^2 would have to be calculated by means of (72), and one question would be to find how far α might be reduced without interfering with the prescribed character of k^2. But to discuss this in detail would lead us too far.

If the differences of quality in the variable medium are small, (72) simplifies. If η_0, k_0 be corresponding values, subject to $k_0^2 = C^4\eta_0^{-4}$, we may take

$$\eta = \eta_0 + \eta', \quad k^2 = k_0^2 + \delta k^2, \quad \text{......................(88)}$$

where η' and δk^2 are small, and (72) becomes approximately

$$\frac{d^2\eta'}{dx^2} + 4k_0^2\eta' = -\eta_0\delta k^2. \quad \text{..........................(89)}$$

Replacing x by t, representing *time*, we see that the problem is the same as that of a pendulum upon which displacing forces act; see *Theory of Sound*, § 66. The analogue of the transition from one uniform medium to another is that of the pendulum initially at rest in the position of equilibrium, upon which at a certain time a displacing force acts. The force may be variable at first, but ultimately assumes a constant value. If there is to be no reflection in the original problem, the force must be of such a character that when it becomes constant the pendulum is left *at rest* in the new position. If the object be to effect the transition between the two states in the shortest possible time, but with forces which are restricted never to exceed the final value, it is pretty evident that the force must

immediately assume the maximum admissible value, and retain it for such a time that the pendulum, then left free, will just reach the new position of equilibrium, after which the force is reimposed. The present solution is excluded, if it be required that the force never decrease in value. Under this restriction the best we can do is to make the force assume at once *half* its final value, and remain constant for a time equal to one-half of the free period. Under this force the pendulum will just swing out to the new position of equilibrium, where it is held on arrival by doubling the force. These cases have already been considered, but the analogue of the pendulum is instructive.

Kelvin* has shown that the equation of the second order

$$\frac{d}{dx}\left(\frac{1}{P}\frac{dy_1}{dx}\right) = y_1 \quad\text{.........................(90)}$$

can be solved by a machine. It is worth noting that an equation of the form (53) is solved at the same time. In fact, if we make

$$y_1 = \frac{dy_2}{dx}, \qquad Py_2 = \frac{dy_1}{dx}, \quad\text{.......................(91)}$$

we get on elimination either (90) for y_1, or

$$\frac{d^2y_2}{dx^2} = Py_2 \quad\text{.............................(92)}$$

for y_2. Equations (91) are those which express directly the action of the machine.

It now remains to consider more in detail some cases where total reflection occurs. When there is merely a simple transition from one medium (1) to another (2), the transmitted wave is

$$\phi_2 = A_2 e^{-ia_2(x-x_1)} e^{i(ct+by)}. \quad\text{.......................(93)}$$

If there is total reflection, a_2 becomes imaginary, say $-ia_2'$; the transmitted wave is then no longer a wave in the ordinary sense, but there remains some disturbance, not conveying energy, and rapidly diminishing as we recede from the surface of transition according to the factor $e^{-a_2'(x-x_1)}$. From (2)

$$a_2^2 = \frac{4\pi^2}{\lambda_2^2}\cos^2\theta_2 = \frac{4\pi^2}{\lambda_1^2}\left(\frac{V_1^2}{V_2^2} - \sin^2\theta_1\right),$$

or

$$a_2' = \frac{2\pi}{\lambda_1}\sqrt{(\sin^2\theta_1 - V_1^2/V_2^2)}. \quad\text{.........................(94)}$$

It appears that soon after the critical angle is passed, the disturbance in the second medium extends sensibly to a distance of only a few wave-lengths.

The circumstances of total reflection at a sudden transition are thus very simple; but total reflection itself does not require a sudden transition, and

* *Roy. Soc. Proc.* 1876, Vol. XXIV. p. 269.

takes place however gradual the passage may be from the first medium to the second, the only condition being that when the second is reached the angle of refraction becomes imaginary. From this point of view total reflection is more naturally regarded as a sort of refraction, reflection proper depending on some degree of abruptness of transition. Phenomena of this kind are familiar in Optics under the name of *mirage*.

In the province of acoustics the vagaries of fog-signals are naturally referred to irregular refraction and reflection in the atmosphere, due to temperature or wind differences; but the difficulty of verifying a suggested explanation on these lines is usually serious, owing to our ignorance of the state of affairs overhead*.

The penetration of vibrations into a medium where no regular waves can be propagated is a matter of considerable interest; but, so far as I am aware, there is no discussion of such a case, beyond that already sketched, relating to a sudden transition between two uniform media. It might have been supposed that oblique propagation through a variable medium would involve too many difficulties, but we have already had opportunity to see that, in reality, obliquity need not add appreciably to the complication of the problem.

To fix ideas, let us suppose that we are dealing with waves in a membrane uniformly stretched with tension T, and of superficial density ρ, which is a function of x only. The equation of vibration is (*Theory of Sound*, § 194)

$$T\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right)=\rho\,\frac{d^2\phi}{dt^2},$$

or, if ϕ be proportional to $e^{i(ct+by)}$, as in (1),

$$\frac{d^2\phi}{dx^2}+(c^2\rho/T-b^2)\,\phi=0, \qquad (95)$$

agreeing with (53) if

$$k^2=c^2\rho/T-b^2. \qquad (96)$$

The waves originally move towards the less dense parts, and total reflection will ensue when a place is reached, at and after which k^2 is negative. The case which best lends itself to analytical treatment is when ρ is a linear function of x. k^2 is then also a linear function; and, by suitable choice of the origin and scale of x, (95) takes the form

$$\frac{d^2\phi}{dx^2}+\tfrac{1}{3}x\phi=0. \qquad (97)$$

* An observation during the exceptionally hot weather of last summer recalled my attention to this subject. A train passing at high speed at a distance of not more than 150 yards was almost inaudible. The wheels were in full view, but the situation was such that the line of vision passed for most of its length pretty close to the highly heated ground. It seemed clear that the sound rays which should have reached the observers were deflected upwards over their heads, which were left in a kind of shadow.

The waves are now supposed to come from the positive side and are totally reflected at $x=0$. The coefficient and sign of x are chosen so as to suit the formulæ about to be quoted.

The solution of (97), appropriate to the present problem, is exactly the integral investigated by Airy to express the intensity of light in the neighbourhood of a caustic*. The line $x=0$ is, in fact, a caustic in the optical sense, being touched by all the *rays*. Airy's integral is

$$W=\int_0^\infty \cos \tfrac{1}{2}\pi (w^3 - mw)\, dw. \quad \text{.....................(98)}$$

It was shown by Stokes† to satisfy (97), if

$$x \text{ (in his notation } n) = (\tfrac{1}{2}\pi)^{2/3}\, m. \quad \text{..................(99)}$$

Calculating by quadratures and from series proceeding by ascending powers of m, Airy tabulated W for values of m lying between $m=\pm 5{\cdot}6$. For larger numerical values of m another method is necessary, for which Stokes gave the necessary formulæ. Writing

$$\phi\ddagger = 2(\tfrac{1}{3}x)^{3/2} = \pi(\tfrac{1}{3}m)^{3/2}, \quad \text{.....................(100)}$$

where the numerical values of m and x are supposed to be taken when these quantities are negative, he found when m is positive

$$W = 2^{\frac{1}{2}}(3m)^{-\frac{1}{4}}\{R\cos(\phi - \tfrac{1}{4}\pi) + S\sin(\phi - \tfrac{1}{4}\pi)\}, \quad \text{.........(101)}$$

where $$R = 1 - \frac{1.5.7.11}{1.2(72\phi)^2} + \frac{1.5.7.11.13.17.19.23}{1.2.3.4(72\phi)^4} - \ldots, \quad \text{...(102)}$$

$$S = \frac{1.5}{1.72\phi} - \frac{1.5.7.11.13.17}{1.2.3(72\phi)^3} + \ldots. \quad \text{.....................(103)}$$

When m is negative, so that W is the integral expressed by writing $-m$ for m in (98),

$$W = 2^{-\frac{1}{2}}(3m)^{-\frac{1}{4}}e^{-\phi}\left\{1 - \frac{1.5}{1.72\phi} + \frac{1.5.7.11}{1.2(72\phi)^2} - \ldots\right\}. \quad \text{......(104)}$$

The first form (101) is evidently fluctuating. The roots of $W \doteq 0$ are given by

$$\phi/\pi = i - 0{\cdot}25 + \frac{0{\cdot}028145}{4i-1} - \frac{0{\cdot}026510}{(4i-1)^3} + \ldots, \quad \text{.........(105)}$$

i being a positive integer, so that for $i=2, 3, 4$, etc., we get

$$m = 4{\cdot}3631,\ 5{\cdot}8922,\ 7{\cdot}2436,\ 8{\cdot}4788, \text{ etc.}$$

For $i=1$, Airy's calculation gave $m=2{\cdot}4955$.

* *Camb. Phil. Trans.* 1838, Vol. VI. p. 379; 1849, Vol. VIII. p. 595.

† *Camb. Phil. Trans.* 1850, Vol. IX.; *Math. and Phys. Papers*, Vol. II. p. 328.

‡ Here used in another sense.

The complete solution of (97) in series of ascending powers of x is to be obtained in the usual way, and the arbitrary constants are readily determined by comparison with (98). Lommel* showed that these series are expressible by means of the Bessel's functions $J_{\frac{1}{3}}$, $J_{-\frac{1}{3}}$. The connection between the complete solutions of (97), as expressed by ascending or by descending semi-convergent series, is investigated in a second memoir by Stokes†. A reproduction of the most important part of Airy's table will be found in Mascart's *Optics* (Vol. I. p. 397).

As total reflection requires, the waves in our problem are stationary as regards x. The realized solution of (95) may be written

$$\phi = W \cos (ct + by), \quad \text{......................(106)}$$

W being the function of x already discussed. On the negative side, when x numerically exceeds a moderate value, the disturbance becomes insensible.

* *Studien über die Bessel'schen Functionen*, Leipzig, 1868.

† *Camb. Phil. Trans.* 1857, Vol. X. p. 106.

361.

SPECTROSCOPIC METHODS.

[*Nature*, Vol. LXXXVIII. p. 377, 1912.]

IN his interesting address on spectroscopic methods, Prof. Michelson falls into a not uncommon error when he says that, in order to obtain a pure spectrum, "two important modifications must be made in Newton's arrangement. First, the light must be allowed to pass through a very *narrow* aperture, and, secondly, a sharp *image* of this aperture must be formed by a lens or mirror."

Both these modifications were made by Newton himself, and with a clear understanding of their advantages. In *Opticks*, Exper. 11, we read:—"In the Sun's Light let into my darkened Chamber through a small round hole in my Window—shut, at about 10 or 12 feet from the Window, I placed a Lens, by which the Image of the hole might be distinctly cast upon a sheet of white Paper, placed at the distance of six, eight, ten, or twelve Feet from the Lens....For in this case the circular Images of the hole which comprise that Image...were terminated most distinctly without any Penumbra, and therefore extended into one another the least that they could, and by consequence the mixture of the Heterogeneous Rays was now the least of all."

And further on:—

"Yet instead of the circular hole F, 'tis better to substitute an oblong hole shaped like a long Parallelogram with its length Parallel to the Prism ABC. For if this hole be an Inch or two long, and but a tenth or twentieth part of an Inch broad or narrower: the Light of the Image *pt* will be as Simple as before or simpler [*i.e.* as compared with a correspondingly narrow circular hole], and the Image will become much broader, and therefore more fit to have Experiments tried in its Light than before."

Again, it was not Bunsen and Kirchhoff who first introduced the collimator into the spectroscope. Swan employed it in 1847, and fully described its use in *Edin. Trans.* Vol. XVI. p. 375, 1849. See also *Edin. Trans.* Vol. XXI. p. 411, 1857; *Pogg. Ann.* C, p. 306, 1857.

These are very minor matters as compared with what Prof. Michelson has to tell of his own achievements and experiences, but it seems desirable that they should be set right.

362.

ON DEPARTURES FROM FRESNEL'S LAWS OF REFLEXION.

[*Philosophical Magazine*, Vol. XXIII. pp. 431—439, 1912.]

In the summer of 1907, in connexion with my experiments upon reflexion from glass at the polarizing angle*, I made observations also upon the diamond, a subject in which Kelvin had expressed an interest. It was known from the work of Jamin and others that the polarization of light reflected from this substance is very far from complete at any angle of incidence, and my first experiments were directed to ascertain whether this irregularity could be plausibly attributed to superficial films of foreign matter, such as so greatly influence the corresponding phenomena in the case of water†. The arrangements were of the simplest. The light from a paraffin flame seen edgeways was reflected from the diamond and examined with a nicol, the angle being varied until the reflexion was a minimum.

In one important respect the diamond offers advantages, in comparison, for instance, with glass, where the surface is the field of rapid chemical changes due presumably to atmospheric influences. On the other hand, the smallness of the available surfaces is an inconvenience which, however, is less felt than it would be, were high precision necessary in the measurements. Two diamonds were employed—one, kindly lent me by Sir W. Crookes, mounted at the end of a bar of lead, the other belonging to a lady's ring. No particular difference in behaviour revealed itself.

The results of repeated observations seemed to leave it improbable that any process of cleaning would do more than reduce the reflexion at the polarizing angle. Potent chemicals, such as hot chromic acid, may be employed, but there is usually a little difficulty in the subsequent preparation. After copious rinsing, at first under the tap and then with distilled water from a wash-bottle, the question arises how to dry the surface. Any ordinary wiping may be expected to nullify the chemical treatment; but if

* *Phil. Mag.* Vol. XVI. p. 444 (1908); *Scientific Papers*, Vol. V. p. 489.

† *Phil. Mag.* Vol. XXXIII. p. 1 (1892); *Scientific Papers*, Vol. III. p. 496.

drops are allowed to dry on, the effect is usually bad. Sometimes it is possible to shake the drops away sufficiently. After a successful operation of this sort wiping with an ordinarily clean cloth usually increases the minimum reflexion, and of course a touch with the finger, however prepared, is much worse. As the result of numerous trials I got the impression that the reflexion could not be reduced below a certain standard which left the flame still easily visible. Rotation of the diamond surface in its own plane seemed to be without effect.

During the last few months I have resumed these observations, using the same diamonds, but with such additions to the apparatus as are necessary for obtaining measures of the residual reflexion. Besides the polarizing nicol, there is required a quarter-wave mica plate and an analysing nicol, to be traversed successively by the light after reflexion, as described in my former papers. The analysing nicol is set alternately at angles $\beta = \pm 45°$. At each of these angles extinction may be obtained by a suitable rotation of the polarizing nicol; and the observation consists in determining the angle $\alpha' - \alpha$ between the two positions. Jamin's k, representing the ratio of reflected amplitudes for the two principal planes when light incident at the angle $\tan^{-1}\mu$ is polarized at 45° to these planes, is equal to $\tan \frac{1}{2}(\alpha' - \alpha)$. The sign of $\alpha' - \alpha$ is reversed when the mica is rotated through a right angle, and the absolute sign of k must be found independently.

Wiped with an ordinarily clean cloth, the diamond gave at first $\alpha' - \alpha = 2°{\cdot}3$. By various treatments this angle could be much reduced. There was no difficulty in getting down to 1°. On the whole the best results were obtained when the surface was finally wiped, or rather pressed repeatedly, upon sheet asbestos which had been ignited a few minutes earlier in the blowpipe flame; but they were not very consistent. The lowest reading was 0°·4; and we may, I think, conclude that with a clean surface $\alpha' - \alpha$ would not exceed 0°·5. No more than in the case of glass, did the effect seem sensitive to moisture, no appreciable difference being observable when chemically dried air played upon the surface. It is impossible to attain absolute certainty, but my impression is that the angle cannot be reduced much further. So long as it exceeds a few tenths of a degree, the paraffin flame is quite adequate as a source of light.

If we take for diamond $\alpha' - \alpha = 30'$, we get

$$k = \tan \tfrac{1}{2}(\alpha' - \alpha) = {\cdot}0044.$$

Jamin's value for k is ·019, corresponding more nearly with what I found for a merely wiped surface.

Similar observations have been made upon the face of a small dispersing prism which has been in my possession some 45 years. When first examined, it gave $\alpha' - \alpha = 9°$, or thereabouts. Treatment with rouge on a piece of

calico, stretched over a glass plate, soon reduced the angle to 4° or 3°, but further progress seemed more difficult. Comparisons were rendered somewhat uncertain by the fact that different parts of the surface gave varying numbers. After a good deal of rubbing, $\alpha' - \alpha$ was reduced to such figures as 2°, on one occasion apparently to $1\frac{1}{2}$°. Sometimes the readings were taken without touching the surface after removal from the rouge, at others the face was breathed upon and wiped. In general, the latter treatment seemed to increase the angle. Strong sulphuric acid was also tried, but without advantage, as also putty-powder in place of or in addition to rouge. The behaviour did not appear to be sensitive to moisture, or to alter appreciably when the surface stood for a few days after treatment.

Thinking that possibly changes due to atmospheric influences might in nearly half a century have penetrated somewhat deeply into the glass, I re-ground and polished (sufficiently for the purpose) one of the originally unpolished faces of the prism, but failed even with this surface to reduce $\alpha' - \alpha$ below 2°. As in the case of the diamond, it is impossible to prove absolutely that $\alpha' - \alpha$ cannot be reduced to zero, but after repeated trials I had to despair of doing so. It may be well to record that the refractive index of the glass for yellow rays is 1·680.

These results, in which k (presumably positive) remained large in spite of all treatment, contrast remarkably with those formerly obtained on less refractive glasses, one of which, however, appears to contain lead. It was then found that by re-polishing it was possible to carry k down to zero and to the negative side, somewhat as in the observations upon water it was possible to convert the negative k of ordinary (greasy) water into one with a small positive value, when the surface was purified to the utmost.

There is another departure from Fresnel's laws which is observed when a piece of plate glass is immersed in a liquid of equal index*. Under such circumstances the reflexion ought to vanish.

The liquid may consist of benzole and bisulphide of carbon, of which the first is less and the second more refractive than the glass. If the adjustment is for the yellow, more benzole or a higher temperature will take the ray of equal index towards the blue and *vice versâ*. "For a closer examination the plate was roughened behind (to destroy the second reflexion), and was mounted in a bottle prism in such a manner that the incidence could be rendered grazing. When the adjustment of indices was for the yellow the appearances observed were as follows: if the incidence is pretty oblique, the reflexion is total for the violet and blue; scanty, but not evanescent, for the yellow; more copious again in the red. As the incidence becomes more and more nearly grazing, the region of total reflexion advances from the blue

* "On the Existence of Reflexion when the relative Refractive Index is Unity," *Brit. Assoc. Report*, p. 585 (1887); *Scientific Papers*, Vol. III. p. 15.

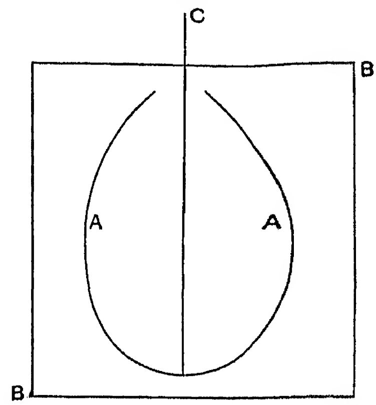
end closer and closer upon the ray of equal index, and ultimately there is a very sharp transition between this region and the band which now looks very dark. On the other side the reflexion revives, but more gradually, and becomes very copious in the orange and red. On this side the reflexion is not technically total. If the prism be now turned so that the angle of incidence is moderate, it is found that, in spite of the equality of index for the most luminous part of the spectrum, there is a pretty strong reflexion of a candle-flame, and apparently without colour. With the aid of sunlight it was proved that in the reflexion at moderate incidences there was no marked chromatic selection, and in all probability the blackness of the band in the yellow at grazing incidences is a matter of contrast only. Indeed, calculation shows that according to Fresnel's formulæ, the reflexion would be nearly insensible for all parts of the spectrum when the index is adjusted for the yellow." It was further shown that the reflexion could be reduced, but not destroyed, by re-polishing or treatment of the surface with hydrofluoric acid.

I have lately thought it desirable to return to these experiments under the impression that formerly I may not have been sufficiently alive to the irregular behaviour of glass surfaces which are in contact with the atmosphere. I wished also to be able to observe the transmitted as well as the reflected light. A cell was prepared from a tin-plate cylinder 3 inches long and 2 inches in diameter by closing the ends with glass plates cemented on with glue and treacle. Within was the glass plate to be experimented on, of similar dimensions, so as to be nearly a fit. A hole in the cylindrical wall allowed the liquid to be poured in and out. Although the plate looked good and had been well wiped, I was unable to reproduce the old effects; or, for a time, even to satisfy myself that I could attain the right composition of the liquid. Afterwards a clue was found in the spectra formed by the edges of the plate (acting as prisms) when the cell was slewed round. The subject of observation was a candle placed at a moderate distance. When the adjustment of indices is correct for any ray, the corresponding part of the spectrum is seen in the same direction as is the undispersed candle-flame by rays which have passed outside the plate. Either spectrum may be used, but the best for the purpose is that formed by the edge nearer the eye. There was now no difficulty in adjusting the index for the yellow ray, and the old effects ought to have manifested themselves; but they did not. The reflected image showed little deficiency in the yellow, although the incidence was nearly grazing, while at moderate angles it was fairly bright and without colour. This considerable departure from Fresnel's laws could only be attributed to a not very thin superficial modification of the glass rendering it optically different from the interior.

In order to allow of the more easy removal and replacement of the plate under examination, an altered arrangement was introduced, in which the

aperture at the top of the cell extended over the whole length. The general dimensions being the same as before, the body of the cell was formed by bending round a rectangular piece of tin-plate A (fig. 1) and securing the ends, to which the glass faces B were to be cemented, by enveloping copper wire. The plate C could then be removed for cleaning or polishing without breaking a joint. In emptying the cell it is necessary to employ a large funnel, as the liquid pours badly.

Fig. 1.



The plate tried behaved much as the one just spoken of. In the reflected light, whether at moderate angles or nearly grazing, the yellow-green ray of equal index did not appear to be missing. A line or rather band of polish, by putty-powder applied with the finger, showed a great alteration. Near grazing there was now a dark band in the spectrum of the reflected light as formerly described, and the effect was intensified when the polish affected *both* faces. In the transmitted light the spectrum was shorn of blue and green, the limit coming down as grazing is approached—a consequence of the total reflexion of certain rays which then sets in. But at incidences far removed from grazing the place of equal index in the spectrum of the reflected light showed little weakening. A few days' standing (after polishing) in the air did not appear to alter the behaviour materially. On the same plate other bands were treated with hydrofluoric acid—commercial acid diluted to one-third. This seemed more effective than the putty-powder. At about 15° off grazing, the spectrum of the reflected light still showed some weakening in the ray of equal index.

In the cell with parallel faces it is not possible to reduce the angle of incidence (reckoned from the normal) sufficiently, a circumstance which led me to revert to the 60° bottle-prism. A strip of glass half an inch wide could be inserted through the neck, and this width suffices for the observation of the reflected light. But I experienced some trouble in finding the light until I had made a calculation of the angles concerned. Supposing the plane of the reflecting surface to be parallel to the base of the prism, let us call the angle of incidence upon it χ, and let θ, ϕ be the angles which the ray makes with the normal to the faces, externally and internally, measured in each case *towards* the refracting angle of the prism. Then

$$\chi = 60^\circ - \phi, \quad \phi = \sin^{-1}(\tfrac{2}{3}\sin\theta).$$

The smallest χ occurs when $\theta = 90^\circ$, in which case $\chi = 18^\circ\ 10'$. This value cannot be actually attained, since the emergence would be grazing. If $\chi = 90^\circ$, giving grazing reflexion, $\theta = -48^\circ\ 36'$. Again, if $\theta = 0$, $\chi = 60^\circ$;

in which K, μ are the electric and magnetic constants for the first medium, K_1, μ_1 for the second*. The relation between θ and θ_1 is

$$K_1\mu_1 : K\mu = \sin^2\theta : \sin^2\theta_1. \quad \text{(C)}$$

It is evident that mere absence of refraction will not secure the evanescence of reflexion for both polarizations, unless we assume *both* $\mu_1 = \mu$ and $K_1 = K$. In the usual theory μ_1 is supposed equal to μ in all cases. (A) then identifies itself with Fresnel's sine-formula, and (B) with the tangent-formula, and both vanish when $K_1 = K$ corresponding to no refraction. Further, (B) vanishes at the Brewsterian angle, even though there be refraction. A slight departure from these laws would easily be accounted for by a difference between μ_1 and μ, such as in fact occurs in some degree (diamagnetism). But the effect of such a departure is not to interfere with the complete evanescence of (B), but merely to displace the angle at which it occurs from the Brewsterian value. If $\mu_1/\mu = 1 + h$, where h is small, calculation shows that the angle of complete polarization is changed by the amount

$$\delta\theta = -\frac{hn^3}{(n^2+1)(n^2-1)}, \quad \text{(D)}$$

n being the refractive index. The failure of the diamond and dense glass to polarize completely at some angle of incidence is not to be explained in this way.

As I formerly suggested, the anomalies may perhaps be connected with the fact that one at least of the media is *dispersive*. A good deal depends upon the cause of the dispersion. In the case of a stretched string, vibrating transversely and endowed with a moderate amount of stiffness, the boundary conditions would certainly be such as would entail a reflexion in spite of equal velocity of wave-propagation. All optical dispersion is now supposed to be of the same nature as what used to be called *anomalous* dispersion, *i.e.* to be due to resonances lying beyond the visible range. In the simplest form of this theory, as given by Maxwell† and Sellmeier, the resonating bodies take their motion from those parts of the æther with which they are directly connected, but they do not influence one another. In such a case the boundary conditions involve merely the continuity of the displacement and its first derivative, and no complication ensues. When there is no refraction, there is also no reflexion. By introducing a mutual reaction between the resonators, and probably in other ways, it would be possible to modify the situation in such a manner that the boundary conditions would involve higher derivatives, as in the case of the stiff string, and thus to allow reflexion in spite of equality of wave-velocities for a given ray.

* "On the Electromagnetic Theory of Light," *Phil. Mag.* Vol. XII. p. 81 (1881); *Scientific Papers*, Vol. I. p. 521.

† *Cambridge Calendar* for 1869. See *Phil. Mag.* Vol. XLVIII. p. 151 (1899); *Scientific Papers*, Vol. IV. p. 413.

and if $\chi = 45°$, $\theta = 22° \ 51'$. We can thus deal with all kinds of reflexion from $\chi = 90°$ down to nearly 18°, and this suffices for the purpose.

The strip employed was of plate glass and was ground upon the back surface. The front reflecting face was treated for about 30^s with hydrofluoric acid. It was now easy to trace the effects all the way from grazing incidence down to an incidence of 45° or less. The ray of equal index was in the yellow-green, as was apparent at once from the spectrum of the reflected light near grazing. There was a very dark band in this region, and total reflexion reaching nearly down to it from the blue end. The light was from a paraffin flame, at a distance of about two feet, seen edgeways. As grazing incidence is departed from, the flame continues at first to show a purple colour, and the spectrum shows a weakened, but not totally absent, green. As the angle of incidence χ still further decreases, the reflected light weakens both in intensity and colour. When $\chi = 45°$, or thereabouts, the light was weak and the colour imperceptible. After two further treatments with hydrofluoric acid and immediate examination, the light seemed further diminished, but it remained bright enough to allow the absence of colour to be ascertained, especially when the lamp was temporarily brought nearer. An ordinary candle-flame at the same (2 feet) distance was easily visible.

In order to allow the use of the stopper, the strip was removed from the bottle-prism when the observations were concluded, and it stood for four days exposed to the atmosphere. On re-examination it seemed that the reflexion at $\chi = 45°$ had sensibly increased, a conclusion confirmed by a fresh treatment with hydrofluoric acid.

It remains to consider the theoretical bearing of the two anomalies which manifest themselves (i) at the polarizing angle, and (ii) at other angles when both media have the same index, at any rate for a particular ray. Evidently the cause may lie in a skin due either to contamination or to the inevitable differences which must occur in the neighbourhood of the surface of a solid or fluid body. Such a skin would explain both anomalies and is certainly a part of the true explanation, but it remains doubtful whether it accounts for everything. Under these circumstances it seems worth while to inquire what would be the effect of less simple boundary conditions than those which lead to Fresnel's formulæ.

On the electromagnetic theory, if θ, θ_1 are respectively the angles of incidence and refraction, the ratio of the reflected to the incident vibration is, for the two principal polarizations,

$$\frac{\tan\theta_1/\tan\theta - \mu/\mu_1}{\tan\theta_1/\tan\theta + \mu/\mu_1} \quad \text{.........................(A)}$$

and

$$\frac{\tan\theta_1/\tan\theta - K/K_1}{\tan\theta_1/\tan\theta + K/K_1}, \quad \text{.........................(B)}$$

P.S. Jan. 15.—Some later observations upon a surface of *fused quartz* are of interest. The plate, prepared by Messrs Hilger, was $\frac{1}{2}$ inch square, and the surfaces were inclined at a few degrees so as to separate the reflexions. From these surfaces the reflexion at the polarizing angle *sensibly disappears.* The image of the paraffin flame could be quenched by the operation of the polarizing nicol alone. When the quarter wave-plate and analysing nicol were introduced, α' and α could not be distinguished, the difference probably not exceeding $\cdot 05^\circ$, *i.e.* 3 minutes of angle.

In order to examine the reflexion when the quartz was in contact with a liquid of equal index, I had to mix alcohol with the benzole. The behaviour was then much the same as with glass of which the surface had been renewed by hydrofluoric acid. No precise measures could be taken, but the reflexion at 45° incidence seemed less than from the glass, though still easily visible. In spite of repeated trials with intermediate cleanings, it was difficult to feel sure that the residual effect might not be due to foreign matter, the more as differences could sometimes be detected between various parts of the surface*. Even if the surface could be regarded as clean on immersion, there is no certainty that a capillary film of some sort might not be deposited upon it from the liquid. The cause of the small residual reflexion must remain for the present an open question.

* At the top of the plate, where it was attached to a handle, a slight invasion of gelatine (used as a cement) gave rise to a copious reflexion; but this film was easily visible in the air.

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THE PRINCIPLE OF REFLECTION IN SPECTROSCOPES.

[*Nature*, Vol. LXXXIX. p. 167, 1912.]

THE application of a reflector to pass light back through a prism, or prisms, is usually ascribed to Littrow. Thus Kayser writes (*Handbuch der Spectroscopie*, Bd. I. p. 513), "Der Erste, der Rückkehr der Strahlen zur Steigerung der Dispersïon verwandte, war Littrow" (O. v. Littrow, *Wien. Ber.* XLVII. ii. pp. 26–32, 1863). But this was certainly not the first use of the method. I learned it myself from Maxwell (*Phil. Trans.* Vol. CL. p. 78, 1860), who says, "The principle of reflecting light, so as to pass twice through the same prism, was employed by me in an instrument for combining colours made in 1856, and a reflecting instrument for observing the spectrum has been constructed by M. Porro."

I have not been able to find the reference to Porro; but it would seem that both Maxwell and Porro antedated Littrow. As to the advantages of the method there can be no doubt.

ON THE SELF-INDUCTION OF ELECTRIC CURRENTS IN A THIN ANCHOR-RING.

[*Proceedings of the Royal Society*, A, Vol. LXXXVI. pp. 562—571, 1912.]

IN their useful compendium of "Formulæ and Tables for the Calculation of Mutual and Self-Inductance*," Rosa and Cohen remark upon a small discrepancy in the formulæ given by myself† and by M. Wien‡ for the self-induction of a coil of circular cross-section over which the current is *uniformly distributed*. With omission of n, representative of the number of windings, my formula was

$$L = 4\pi a\left[\log\frac{8a}{\rho} - \frac{7}{4} + \frac{\rho^2}{8a^2}\left(\log\frac{8a}{\rho} + \frac{1}{3}\right)\right], \quad \text{..............(1)}$$

where ρ is the radius of the section and a that of the circular axis. The first two terms were given long before by Kirchhoff§. In place of the fourth term within the bracket, viz., $+\frac{1}{24}\rho^2/a^2$, Wien found $-\cdot 0083\rho^2/a^2$. In either case a correction would be necessary in practice to take account of the space occupied by the insulation. Without, so far as I see, giving a reason, Rosa and Cohen express a preference for Wien's number. The difference is of no great importance, but I have thought it worth while to repeat the calculation and I obtain the same result as in 1881. A confirmation after 30 years, and without reference to notes, is perhaps almost as good as if it were independent. I propose to exhibit the main steps of the calculation and to make extension to some related problems.

The starting point is the expression given by Maxwell ‖ for the mutual induction M between two neighbouring co-axial circuits. For the present

* *Bulletin of the Bureau of Standards*, Washington, 1908, Vol. III. No. 1.

† *Roy. Soc. Proc.* 1881, Vol. XXXII. p. 104; *Scientific Papers*, Vol. II. p. 15.

‡ *Ann. d. Physik*, 1894, Vol. LIII. p. 934; it would appear that Wien did not know of my earlier calculation.

§ *Pogg. Ann.* 1864, Vol. CXXI. p. 551.

‖ *Electricity and Magnetism*, § 705.

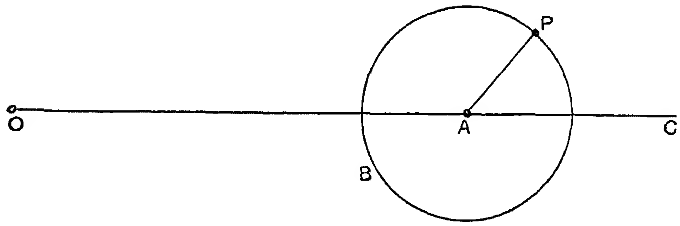
purpose this requires transformation, so as to express the inductance in terms of the situation of the elementary circuits relatively to the circular axis. In the figure, O is the centre of the circular axis, A the centre of a section B through the axis of symmetry, and the position of any point P of the section is given by polar coordinates relatively to A, viz., by PA (ρ) and by the angle $PAC(\phi)$. If ρ_1, ϕ_1; ρ_2, ϕ_2 be the coordinates of two points of the section P_1, P_2, the mutual induction between the two circular circuits represented by P_1, P_2 is approximately

$$\begin{aligned}\frac{M_{12}}{4\pi a} = &\left\{1 + \frac{\rho_1 \cos\phi_1 + \rho_2 \cos\phi_2}{2a} + \frac{\rho_1^2 + \rho_2^2 + 2\rho_1^2 \sin^2\phi_1 + 2\rho_2^2 \sin^2\phi_2}{16a^2}\right.\\ &\left.- \frac{2\rho_1\rho_2 \cos(\phi_1 - \phi_2) + 4\rho_1\rho_2 \sin\phi_1 \sin\phi_2}{16a^2}\right\} \log\frac{8a}{r}\\ &- 2 - \frac{\rho_1 \cos\phi_1 + \rho_2 \cos\phi_2}{2a}\\ &+ \frac{3(\rho_1^2 + \rho_2^2) - 4(\rho_1^2 \sin^2\phi_1 + \rho_2^2 \sin^2\phi_2) + 2\rho_1\rho_2 \cos(\phi_1 - \phi_2)}{16a^2},\end{aligned} \quad (2)$$

in which r, the distance between P_1 and P_2, is given by

$$r^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\phi_1 - \phi_2). \quad \text{.....................(3)}$$

Further details will be found in Wien's memoir; I do not repeat them because I am in complete agreement so far.



For the problem of a current uniformly distributed we are to integrate (2) twice over the area of the section. Taking first the integrations with respect to ϕ_1, ϕ_2, let us express

$$\frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} \frac{M_{12}}{4\pi a}\, d\phi_1\, d\phi_2, \quad \text{.........................(4)}$$

of which we can also make another application. The integration of the terms which do not involve $\log r$ is elementary. For those which do involve $\log r$ we may conveniently replace ϕ_2 by $\phi_1 + \phi$, where $\phi = \phi_2 - \phi_1$, and take first the integration with respect to ϕ, ϕ_1 being constant. Subsequently we integrate with respect to ϕ_1.

It is evident that the terms in (2) which involve the first power of ρ vanish in the integration. For a change of ϕ_1, ϕ_2 into $\pi - \phi_1$, $\pi - \phi_2$

respectively reverses $\cos\phi_1$ and $\cos\phi_2$, while it leaves r unaltered. The definite integrals required for the other terms are*

$$\int_{-\pi}^{+\pi} \log(\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\phi)\,d\phi = \text{greater of } 4\pi\log\rho_2 \text{ and } 4\pi\log\rho_1, \quad (5)$$

$$\int_{-\pi}^{+\pi} \cos m\phi \log(\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\phi)\,d\phi$$
$$= -\frac{2\pi}{m} \times \text{smaller of } \left(\frac{\rho_2}{\rho_1}\right)^m \text{ and } \left(\frac{\rho_1}{\rho_2}\right)^m, \quad \ldots\ldots(6)$$

m being an integer. Thus

$$\frac{1}{4\pi^2}\iint \log r\,d\phi_1\,d\phi_2 = \frac{1}{4\pi^2}\int_{-\pi}^{+\pi} d\phi_1 \int_{-\pi}^{+\pi} d\phi \log r = \text{greater of } \log\rho_2 \text{ and } \log\rho_1. \quad (7)$$

So far as the more important terms in (4)—those which do not involve ρ as a factor—we have at once

$$\log(8a) - 2 - \text{greater of } \log\rho_2 \text{ and } \log\rho_1. \quad \ldots\ldots\ldots\ldots(8)$$

If ρ_2 and ρ_1 are equal, this becomes

$$\log(8a/\rho) - 2. \quad \ldots\ldots\ldots\ldots(9)$$

We have now to consider the terms of the second order in (2). The contribution which these make to (4) may be divided into two parts. The first, not arising from the terms in $\log r$, is easily found to be

$$\frac{\rho_1^2+\rho_2^2}{8a^2}\left[\log(8a)+\tfrac{1}{2}\right]. \quad \ldots\ldots\ldots\ldots(10)$$

The difference between Wien's number and mine arises from the integration of the terms in $\log r$, so that it is advisable to set out these somewhat in detail. Taking the terms in order, we have as in (7)

$$\frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} \log r\,d\phi_1\,d\phi_2 = \text{greater of } \log\rho_2 \text{ and } \log\rho_1. \quad \ldots\ldots(11)$$

In like manner

$$\frac{1}{4\pi^2}\iint \sin^2\phi_1 \log r\,d\phi_1\,d\phi_2 = \tfrac{1}{2}\left[\text{greater of } \log\rho_2 \text{ and } \log\rho_1\right], \quad \ldots(12)$$

and $\frac{1}{4\pi^2}\iint \sin^2\phi_2 \log r\,d\phi_1\,d\phi_2$ has the same value. Also by (6), with $m=1$,

$$\frac{1}{4\pi^2}\iint \cos(\phi_2-\phi_1)\log r\,d\phi_1 d\phi_2 = -\tfrac{1}{2}\left[\text{smaller of } \rho_2/\rho_1 \text{ and } \rho_1/\rho_2\right]. \ldots(13)$$

Finally $\frac{1}{4\pi^2}\iint \sin\phi_1 \sin\phi_2 \log r\,d\phi_1\,d\phi_2$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{+\pi} d\phi_1 \sin\phi_1 \int_{-\pi}^{+\pi} (\sin\phi_1\cos\phi + \cos\phi_1\sin\phi)\log r\,d\phi$$
$$= -\tfrac{1}{4}\left[\text{smaller of } \rho_2/\rho_1 \text{ and } \rho_1/\rho_2\right]. \quad \ldots\ldots\ldots\ldots(14)$$

* Todhunter's *Int. Calc.* §§ 287, 289.

Thus altogether the terms in (2) of the second order involving $\log r$ yield in (4)

$$-\frac{\rho_1^2+\rho_2^2}{8a^2}[\text{greater of }\log\rho_2\text{ and }\log\rho_1]-\frac{\rho_1\rho_2}{8a^2}\left[\text{smaller of }\frac{\rho_2}{\rho_1}\text{ and }\frac{\rho_1}{\rho_2}\right]. \quad \ldots(15)$$

The complete value of (4) to this order of approximation is found by addition of (8), (10), and (15).

By making ρ_2 and ρ_1 equal we obtain at once for the self-induction of a current limited to the circumference of an anchor-ring, and uniformly distributed over that circumference,

$$L=4\pi a\left[\left(1+\frac{\rho^2}{4a^2}\right)\log\frac{8a}{\rho}-2\right], \quad \ldots\ldots\ldots\ldots(16)$$

ρ being the radius of the circular section. The value of L for this case, when ρ^2 is neglected, was virtually given by Maxwell*.

When the current is uniformly distributed over the area of the section, we have to integrate again with respect to ρ_1 and ρ_2 between the limits 0 and ρ in each case. For the more important terms we have from (8)

$$\frac{1}{\rho^4}\iint d\rho_1^2\,d\rho_2^2\,[\log 8a-2-\text{greater of }\log\rho_2\text{ and }\log\rho_1]$$

$$=\log 8a-2-\frac{1}{2\rho^4}\int d\rho_1^2\,[\log\rho_1^2\,.\,\rho_1^2+\rho^2(\log\rho^2-1)-\rho_1^2(\log\rho_1^2-1)]$$

$$=\log 8a-2-\log\rho+\frac{1}{4}=\log\frac{8a}{\rho}-\frac{7}{4}. \quad \ldots\ldots\ldots\ldots(17)$$

A similar operation performed upon (10) gives

$$\frac{\log(8a)+\frac{1}{2}}{8a^2\rho^4}\iint(\rho_1^2+\rho_2^2)\,d\rho_1^2\,d\rho_2^2=\frac{\log(8a)+\frac{1}{2}}{8}\frac{\rho^2}{a^2}. \quad \ldots\ldots(18)$$

In like manner, the first part of (15) yields

$$-\frac{\rho^2}{16a^2}(\log\rho^2-\tfrac{1}{3}).$$

For the second part we have

$$-\frac{1}{8a^2\rho^4}\iint d\rho_1^2 d\rho_2^2\,[\text{smaller of }\rho_2^2,\,\rho_1^2]=-\frac{\rho^2}{24a^2};$$

thus altogether from (15)

$$-\frac{\rho^2}{8a^2}(\log\rho+\tfrac{1}{6}). \quad \ldots\ldots\ldots\ldots(19)$$

The terms of the second order are accordingly, by addition of (18) and (19),

$$\frac{\rho^2}{8a^2}\left(\log\frac{8a}{\rho}+\frac{1}{3}\right). \quad \ldots\ldots\ldots\ldots(20)$$

* *Electricity and Magnetism*, §§ 692, 706.

To this are to be added the leading terms (17); whence, introducing $4\pi a$, we get finally the expression for L already stated in (1).

It must be clearly understood that the above result, and the corresponding one for a *hollow* anchor-ring, depend upon the assumption of a uniform distribution of current, such as is approximated to when the coil consists of a great number of windings of wire insulated from one another. If the conductor be solid and the currents due to induction, the distribution will, in general, not be uniform. Under this head Wien considers the case where the currents are due to the variation of a homogeneous magnetic field, parallel to the axis of symmetry, and where the distribution of currents is governed by *resistance*, as will happen in practice when the variations are slow enough. In an elementary circuit the electromotive force varies as the square of the radius and the resistance as the first power. Assuming as before that the whole current is unity, we have merely to introduce into (4) the factors

$$\frac{(a+\rho_1\cos\phi_1)(a+\rho_2\cos\phi_2)}{a^2}, \quad \ldots\ldots\ldots\ldots\ldots(21)$$

M_{12} retaining the value given in (2).

The leading term in (21) is unity, and this, when carried into (14), will reproduce the former result. The term of the first order in ρ in (21) is $(\rho_1\cos\phi_1+\rho_2\cos\phi_2)/a$, and this must be combined with the terms of order ρ^0 and ρ^1 in (2). The former, however, contributes nothing to the integral. The latter yield in (4)

$$\frac{\rho_1^2+\rho_2^2}{4a^2}\{\log 8a - 1 - \text{greater of} \log\rho_1 \text{ and } \log\rho_2\} + \frac{\text{smaller of } \rho_1^2 \text{ and } \rho_2^2}{4a^2}. \quad (22)$$

The term of the second order in (21), viz., $\rho_1\rho_2/a^2 . \cos\phi_1\cos\phi_2$, needs to be combined only with the leading term in (2). It yields in (4)

$$\frac{\text{smaller of } \rho_1^2 \text{ and } \rho_2^2}{4a^2}. \quad \ldots\ldots\ldots\ldots\ldots(23)$$

If ρ_1 and ρ_2 are equal (ρ), the additional terms expressed by (22), (23) become

$$\frac{\rho^2}{2a^2}\log\frac{8a}{\rho}. \quad \ldots\ldots\ldots\ldots\ldots(24)$$

If (24), multiplied by $4\pi a$, be added to (16), we shall obtain the self-induction for a shell (of uniform infinitesimal thickness) in the form of an anchor-ring, the currents being excited in the manner supposed. The result is

$$L = 4\pi a\left\{\left(1+\frac{3\rho^2}{4a^2}\right)\log\frac{8a}{\rho} - 2\right\}. \quad \ldots\ldots\ldots\ldots\ldots(25)$$

We now proceed to consider the solid ring. By (22), (23) the terms, additional to those previously obtained on the supposition that the current was uniformly distributed, are

$$\frac{1}{\rho^4}\iint d\rho_1^2\,d\rho_2^2\left[\frac{\text{smaller of }\rho_1^2\text{ and }\rho_2^2}{2a^2}\right.$$
$$\left.+\frac{\rho_1^2+\rho_2^2}{4a^2}\{\log 8a-1-\text{greater of }\log\rho_1\text{ and }\log\rho_2\}\right]. \quad\ldots (26)$$

The first part of this is $\rho^2/6a^2$, and the second is $\frac{\rho^2}{4a^2}\{\log 8a - 1 - \log\rho + \frac{1}{6}\}$. The additional terms are accordingly

$$\frac{\rho^2}{4a^2}\left\{\log\frac{8a}{\rho}-\frac{1}{6}\right\}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

These multiplied by $4\pi a$ are to be added to (1). We thus obtain

$$L = 4\pi a\left[\log\frac{8a}{\rho}-\frac{7}{4}+\frac{3\rho^2}{8a^2}\log\frac{8a}{\rho}\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

for the self-induction of the solid ring when currents are slowly generated in it by uniform magnetic forces parallel to the axis of symmetry. In Wien's result for this case there appears an additional term within the bracket equal to $-0{\cdot}092\,\rho^2/a^2$.

A more interesting problem is that which arises when the alternations in the magnetic field are rapid instead of slow. Ultimately the distribution of current becomes independent of *resistance*, and is determined by induction alone. A leading feature is that the currents are *superficial*, although the ring itself may be solid. They remain, of course, symmetrical with respect to the straight axis, and to the plane which contains the circular axis.

The magnetic field may be supposed to be due to a current x_1 in a circuit at a distance, and the whole energy of the field may be represented by

$$T = \tfrac{1}{2}M_{11}x_1^2 + \tfrac{1}{2}M_{22}x_2^2 + \tfrac{1}{2}M_{33}x_3^2 + \ldots + M_{12}x_1x_2 + M_{13}x_1x_3 + \ldots$$
$$+ M_{23}x_2x_3 + \ldots, \quad\ldots\ldots(29)$$

x_2, x_3, etc., being currents in other circuits where no independent electromotive force acts. If x_1 be regarded as given, the corresponding values of $x_2, x_3, \ldots$ are to be found by making T a minimum. Thus

$$\left.\begin{aligned} M_{12}x_1 + M_{22}x_2 + M_{32}x_3 + \ldots &= 0,\\ M_{13}x_1 + M_{23}x_2 + M_{33}x_3 + \ldots &= 0,\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

and so on, are the equations by which x_2, etc., are to be found in terms of x_1. What we require is the corresponding value of T', formed from T by omission of the terms containing x_1.

The method here sketched is general. It is not necessary that x_2, etc., be currents in particular circuits. They may be regarded as generalized

coordinates, or rather velocities, by which the kinetic energy of the system is defined.

For the present application we suppose that the distribution of current round the circumference of the section is represented by

$$\{\alpha_0 + \alpha_1 \cos \phi_1 + \alpha_2 \cos 2\phi_1 + \ldots\} \frac{d\phi_1}{2\pi}, \quad \ldots\ldots\ldots\ldots\ldots(31)$$

so that the total current is α_0. The doubled energy, so far as it depends upon the interaction of the ring currents, is

$$\frac{1}{4\pi^2}\iint (\alpha_0 + \alpha_1 \cos \phi_1 + \alpha_2 \cos 2\phi_1 + \ldots)(\alpha_0 + \alpha_1 \cos \phi_2 + \ldots) M_{12} d\phi_1 d\phi_2, \quad (32)$$

where M_{12} has the value given in (2), simplified by making ρ_1 and ρ_2 both equal to ρ. To this has to be added the double energy arising from the interaction of the ring currents with the primary current. For each element of the ring currents (31) we have to introduce a factor proportional to the area of the circuit, viz., $\pi(a + \rho \cos \phi_1)^2$. This part of the double energy may thus be taken to be

$$H \int d\phi_1 (a + \rho \cos \phi_1)^2 (\alpha_0 + \alpha_1 \cos \phi_1 + \alpha_2 \cos 2\phi_1 + \ldots),$$

that is
$$2\pi H \{(a^2 + \tfrac{1}{2}\rho^2)\alpha_0 + a\rho\alpha_1 + \tfrac{1}{4}\rho^2\alpha_2\}, \quad \ldots\ldots\ldots\ldots\ldots(33)$$

α_3, etc., not appearing. The sum of (33) and (32) is to be made a minimum by variation of the α's.

We have now to evaluate (32). The coefficient of α_0^2 is the quantity already expressed in (16). For the other terms it is not necessary to go further than the first power of ρ in (2). We get

$$4\pi a \left[\alpha_0^2 \left\{\log \frac{8a}{\rho}\left(1 + \frac{\rho^2}{4a^2}\right) - 2\right\} + \tfrac{1}{4}(\alpha_1^2 + \tfrac{1}{2}\alpha_2^2 + \tfrac{1}{3}\alpha_3^2 + \ldots)\right.$$
$$\left. + \frac{\rho}{a}\left\{\frac{\alpha_0\alpha_1}{2}\left(\log \frac{8a}{\rho} - 1\right) + \frac{\alpha_1}{8}(2\alpha_0 + \alpha_2) + \frac{\alpha_2}{2\,.\,8}(\alpha_1 + \alpha_3) + \frac{\alpha_3}{3\,.\,8}(\alpha_2 + \alpha_4) + \ldots\right\}\right].$$
$$\ldots\ldots\ldots(34)$$

Differentiating the sum of (33), (34), with respect to α_0, α_1, etc., in turn, we find

$$H(a^2 + \tfrac{1}{2}\rho^2) + 4a\alpha_0\left\{\log \frac{8a}{\rho}\left(1 + \frac{\rho^2}{4a^2}\right) - 2\right\} + \rho\alpha_1\left(\log \frac{8a}{\rho} - \frac{1}{2}\right) = 0, \quad (35)$$

$$H\rho + \alpha_1 + \frac{\rho}{a}\left\{\alpha_0\left(\log \frac{8a}{\rho} - \frac{1}{2}\right) + \frac{3\alpha_2}{8}\right\} = 0, \quad \ldots\ldots\ldots\ldots\ldots(36)$$

$$H\rho^2 + 2a\alpha_2 + \rho\left(\frac{3\alpha_1}{2} + \frac{5\alpha_3}{6}\right) = 0. \quad \ldots\ldots\ldots\ldots\ldots(37)$$

The leading term is, of course, α_0. Relatively to this, α_1 is of order ρ, α_2 of order ρ^2, and so on. Accordingly, α_2, α_3, etc., may be omitted entirely from (34), which is only expected to be accurate up to ρ^2 inclusive. Also, in α_1 only the leading term need be retained.

The ratio of α_1 to α_0 is to be found by elimination of H between (35), (36). We get

$$\frac{\alpha_1}{\alpha_0} = \frac{\rho}{a}\left\{3 \log \frac{8a}{\rho} - \frac{15}{2}\right\}. \quad \text{.......................(38)}$$

Substituting this in (34), we find as the coefficient of self-induction

$$L = 4\pi a \left[\log \frac{8a}{\rho}\left(1 + \frac{\rho^2}{4a^2}\right) - 2 + \frac{\rho^2}{4a^2}\left(3 \log \frac{8a}{\rho} - \frac{15}{2}\right)\left(5 \log \frac{8a}{\rho} - \frac{17}{2}\right)\right]. \quad (39)$$

The approximate value of α_0 in terms of H is

$$\alpha_0 = -\frac{Ha}{4\left(\log \frac{8a}{\rho} - 2\right)} \quad \text{.............................(40)}$$

A closer approximation can be found by elimination of α_1 between (35), (36).

In (39) the currents are supposed to be induced by the variation (in time) of an unlimited uniform magnetic field. A problem, simpler from the theoretical point of view, arises if we suppose the uniform field to be limited to a cylindrical space co-axial with the ring, and of diameter less than the smallest diameter of the ring $(2a - 2\rho)$. Such a field may be supposed to be due to a cylindrical current sheet, the length of the cylinder being infinite. The ring currents to be investigated are those arising from the instantaneous abolition of the current sheet and its conductor.

If πb^2 be the area of the cylinder, (33) is replaced simply by

$$H\int d\phi_1\, b^2(\alpha_0 + \alpha_1 \cos\phi_1 + \ldots) = 2\pi H b^2 \alpha_0. \quad \text{..............(41)}$$

The expression (34) remains unaltered and the equations replacing (35), (36) are thus

$$Hb^2 + 4a\alpha_0\left\{\log \frac{8a}{\rho}\left(1 + \frac{\rho^2}{4a^2}\right) - 2\right\} + \rho\alpha_1\left(\log \frac{8a}{\rho} - \frac{1}{2}\right) = 0, \quad \text{....(42)}$$

$$\alpha_1 + \frac{\rho}{a}\left(\log \frac{8a}{\rho} - \frac{1}{2}\right)\alpha_0 = 0. \quad \text{.....................(43)}$$

The introduction of (43) into (34) gives for the coefficient of self-induction in this case—

$$L = 4\pi a\left[\log \frac{8a}{\rho}\left(1 + \frac{\rho^2}{4a^2}\right) - 2 - \frac{\rho^2}{4a^2}\left(\log \frac{8a}{\rho} - \frac{1}{2}\right)^2\right]. \quad \text{.......(44)}$$

It will be observed that the sign of α_1/α_0 is different in (38) and (43).

The peculiarity of the problem last considered is that the primary current occasions no magnetic force at the surface of the ring. The consequences were set out 40 years ago by Maxwell in a passage* whose significance was very slowly appreciated. "In the case of a current sheet of no resistance, the surface integral of magnetic induction remains constant at every point of the current sheet.

"If, therefore, by the motion of magnets or variations of currents in the neighbourhood, the magnetic field is in any way altered, electric currents will be set up in the current sheet, such that their magnetic effect, combined with that of the magnets or currents in the field, will maintain the normal component of magnetic induction at every point of the sheet unchanged. If at first there is no magnetic action, and no currents in the sheet, then the normal component of magnetic induction will always be *zero at every point* of the sheet.

"The sheet may therefore be regarded as impervious to magnetic induction, and the lines of magnetic induction will be deflected by the sheet exactly in the same way as the lines of flow of an electric current in an infinite and uniform conducting mass would be deflected by the introduction of a sheet of the same form made of a substance of infinite resistance.

"If the sheet forms a closed or an infinite surface, no magnetic actions which may take place on one side of the sheet will produce any magnetic effect on the other side."

All that Maxwell says of a current sheet is, of course, applicable to the surface of a perfectly conducting solid, such as our anchor-ring may be supposed to be. The currents left in the ring after the abolition of the primary current must be such that the magnetic force due to them is *wholly tangential* to the surface of the ring. Under this condition $\int_{-\pi}^{+\pi} M_{r_1} d\phi_2$ must be independent of ϕ_1, and we might have investigated the problem upon this basis.

In Maxwell's notation α, β, γ denote the components of magnetic force, and the whole energy of the field T is given by

$$T = \frac{1}{8\pi}\iiint(\alpha^2+\beta^2+\gamma^2)\,dxdydz = \tfrac{1}{2}L\alpha_0^2. \quad \ldots\ldots\ldots\ldots(45)$$

Moreover α_0, the total current, multiplied by 4π is equal to the "circulation" of magnetic force round the ring. In this form our result admits of immediate application to the hydrodynamical problem of the circulation of

* *Electricity and Magnetism*, §§ 654, 655. Compare my "Acoustical Observations," *Phil. Mag.* 1882, Vol. XIII. p. 340; *Scientific Papers*, Vol. II. p. 99.

incompressible frictionless fluid round a solid having the form of the ring; for the components of velocity u, v, w are subject to precisely the same conditions as are α, β, γ. If the density be unity, the kinetic energy T of the motion has the expression

$$T = \frac{L}{8\pi} \times (\text{circulation})^2, \qquad \text{.........................(46)}$$

L having the value given in (44).

P.S. March 4.—Sir W. D. Niven, who in 1881 verified some other results for self-induction—those numbered (11), (12) in the paper referred to—has been good enough to confirm the formulæ (1), (28) of the present communication, in which I differ from M. Wien.

365.

ELECTRICAL VIBRATIONS ON A THIN ANCHOR-RING.

[*Proceedings of the Royal Society*, A, Vol. LXXXVII. pp. 193–202, 1912.]

ALTHOUGH much attention has been bestowed upon the interesting subject of electric oscillations, there are comparatively few examples in which definite mathematical solutions have been gained. These problems are much simplified when conductors are supposed to be perfect, but even then the difficulties usually remain formidable. Apart from cases where the propagation may be regarded as being in one dimension*, we have Sir J. Thomson's solutions for electrical vibrations upon a conducting sphere or cylinder†. But these vibrations have so little persistence as hardly to deserve their name. A more instructive example is afforded by a conductor in the form of a circular ring, whose circular section is supposed small. There is then in the neighbourhood of the conductor a considerable store of energy which is more or less entrapped, and so allows of vibrations of reasonable persistence. This problem was very ably treated by Pocklington‡ in 1897, but with deficient explanations§. Moreover, Pocklington limits his detailed conclusions to one particular mode of free vibration. I think I shall be doing a service in calling attention to this investigation, and in exhibiting the result for the radiation of vibrations in the higher modes. But I do not attempt a complete re-statement of the argument.

Pocklington starts from Hertz's formulæ for an elementary vibrator at the origin of coordinates ξ, η, ζ,

$$P = \frac{d^2\Pi}{d\xi\, d\zeta}, \qquad Q = \frac{d^2\Pi}{d\eta\, d\zeta}, \qquad R = \frac{d^2\Pi}{d\zeta^2} + \alpha^2\Pi, \quad \ldots\ldots\ldots\ldots(1)$$

where

$$\Pi = e^{i\alpha\rho}\, e^{ipt}/\rho, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

* *Phil. Mag.* 1897, Vol. XLIII. p. 125; 1897, Vol. XLIV. p. 199; *Scientific Papers*, Vol. IV. pp. 276, 327.

† *Recent Researches*, 1893, §§ 301, 312. [1913. There is also Abraham's solution for the ellipsoid.]

‡ *Camb. Proceedings*, 1897, Vol. IX. p. 324.

§ Compare W. McF. Orr, *Phil. Mag.* 1903, Vol. VI. p. 667.

in which P, Q, R denote the components of electromotive intensity, $2\pi/p$ is the period of the disturbance, and $2\pi/\alpha$ the wave-length corresponding in free æther to this period. At a great distance ρ from the source, we have from (1)

$$P,\ Q,\ R = \frac{\alpha^2 e^{i\alpha\rho}}{\rho}\left(-\frac{\xi\zeta}{\rho^2},\ -\frac{\eta\zeta}{\rho^2},\ 1-\frac{\zeta^2}{\rho^2}\right). \quad \text{.................(3)}$$

The resultant is perpendicular to ρ, and in the plane containing ρ and ζ. Its magnitude is

$$-\frac{\alpha^2 e^{i\alpha\rho}}{\rho}\sin\chi, \quad \text{................................(4)}$$

where χ is the angle between ρ and ζ.

The required solution is obtained by a distribution of elementary vibrators of this kind along the circular axis of the ring, the axis of the vibrator being everywhere tangential to the axis of the ring and the coefficient of intensity proportional to $\cos m\phi'$, where m is an integer and ϕ' defines a point upon the axis. The calculation proceeds in terms of semi-polar coordinates z, ϖ, ϕ, the axis of symmetry being that of z, and the origin being at the centre of the circular axis. The radius of the circular axis is a, and the radius of the circular section is ϵ, ϵ being very small relatively to a. The condition to be satisfied is that at every point of the surface of the ring, where $(\varpi - a)^2 + z^2 = \epsilon^2$, the tangential component of (P, Q, R) shall vanish. It is not satisfied absolutely by the above specification; but Pocklington shows that to the order of approximation required the specification suffices, provided α be suitably chosen. The equation determining α expresses the evanescence of that tangential component which is parallel to the circular axis, and it takes the form

$$\int_0^\pi d\phi\, \Pi_0 \cos m\phi\,(m^2 - \alpha^2 a^2\cos\phi) = 0, \quad \text{.................(5)}$$

where

$$\Pi_0 = \frac{e^{i\alpha[\epsilon^2 + 4\varpi a \sin^2\frac{1}{2}\phi]}}{\sqrt{[\epsilon^2 + 4\varpi a\sin^2\frac{1}{2}\phi]}}. \quad \text{........................(6)}$$

In (5) we are to retain the large term, arising in the integral when ϕ is small, and the finite term, but we may reject *small* quantities. Thus Pocklington finds

$$\int_0^\pi \frac{(a^2\alpha^2\cos\phi - m^2)\cos m\phi\, d\phi}{\sqrt{\{\epsilon^2 + 4a^2\sin^2\frac{1}{2}\phi\}}}$$

$$+\int_0^\pi \frac{(e^{2ia\alpha\sin\frac{1}{2}\phi} - 1)(a^2\alpha^2\cos\phi - m^2)\cos m\phi\, d\phi}{2a\sin\frac{1}{2}\phi} = 0, \quad \text{.........(7)}$$

the condition being to this order of approximation the same at all points of a cross-section.

The first integral in (7) may be evaluated for any (integral) value of m. Writing $\frac{1}{2}\phi = \psi$, we have

$$\int_0^{\frac{1}{2}\pi} \frac{(a^2\alpha^2 \cos 2\psi - m^2) \cos 2m\psi \, d\psi}{a\sqrt{\{\epsilon^2/4a^2 + \sin^2\psi\}}}. \quad \text{.................(8)}$$

The large part of the integral arises from small values of ψ. We divide the range of integration into two parts, the first from 0 to ψ where ψ, though small, is large compared with $\epsilon/2a$, and the second from ψ to $\frac{1}{2}\pi$. For the first part we may replace $\cos 2\psi$, $\cos 2m\psi$ by unity, and $\sin^2\psi$ by ψ^2. We thus obtain

$$\frac{\alpha^2 a^2 - m^2}{a} \log \{\psi + \sqrt{(\epsilon^2/4a^2 + \psi^2)}\}_0^{\psi} = \frac{\alpha^2 a^2 - m^2}{a} (\log 4a/\epsilon + \log \psi). \ \text{...(9)}$$

Thus to a first approximation $\alpha a = \pm m$. In the second part of the range of integration we may neglect $\epsilon^2/4a^2$ in comparison with $\sin^2\psi$, thus obtaining

$$\int_{\psi}^{\frac{1}{2}\pi} \frac{(a^2\alpha^2 \cos 2\psi - m^2) \cos 2m\psi \, d\psi}{a \sin\psi}. \quad \text{.................(10)}$$

The numerator may be expressed as a sum of terms such as $\cos^{2n}\psi$, and for each of these the integral may be evaluated by taking $\cos\psi = z$, in virtue of

$$\int \frac{z^{2n}\,dz}{z^2 - 1} = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\ldots + \frac{z^{2n-1}}{2n-1} + \tfrac{1}{2} \log \frac{1-z}{1+z}.$$

Accordingly

$$\int_{\psi}^{\frac{1}{2}\pi} \frac{\cos^{2n}\psi \, d\psi}{\sin\psi} = -\cos\psi - \frac{\cos^3\psi}{3} - \ldots\ldots - \frac{\cos^{2n-1}\psi}{2n-1} - \log\tan\tfrac{1}{2}\psi$$

$$= -1 - \tfrac{1}{3} - \ldots\ldots - \frac{1}{2n-1} - \log\tfrac{1}{2}\psi, \quad \text{.................(11)}$$

when small quantities are neglected. For example,

$$\int_{\psi}^{\frac{1}{2}\pi} \frac{\cos^2\psi \, d\psi}{\sin\psi} = -1 - \log\tfrac{1}{2}\psi, \qquad \int_{\psi}^{\frac{1}{2}\pi} \frac{\cos^4\psi \, d\psi}{\sin\psi} = -\tfrac{4}{3} - \log\tfrac{1}{2}\psi.$$

The sum of the coefficients in the series of terms (analogous to $\cos^{2n}\psi$) which represents the numerator of (10) is necessarily $a^2\alpha^2 - m^2$, since this is the value of the numerator itself when $\psi = 0$. The particular value of ψ chosen for the division of the range of integration thus disappears from the sum of (9) and (10), as of course it ought to do.

When $m = 1$, corresponding to the gravest mode of vibration specially considered by Pocklington, the numerator in (10) is

$$4a^2\alpha^2 \cos^4\psi - (4a^2\alpha^2 + 2)\cos^2\psi + a^2\alpha^2 + 1,$$

and the value of the integral is accordingly

$$\frac{1}{a}\left[2 - \frac{4a^2\alpha^2}{3} - (a^2\alpha^2 - 1)\log\tfrac{1}{2}\psi\right].$$

To this is to be added from (9)

$$\frac{a^2\alpha^2 - 1}{a}\left[\log\frac{4a}{\epsilon} + \log\psi\right],$$

making altogether for the value of (8)

$$\frac{1}{a}\left[(a^2\alpha^2 - 1)\log\frac{8a}{\epsilon} + 2 - \frac{4a^2\alpha^2}{3}\right]. \quad \text{.................(12)}$$

The second integral in (7) contributes only finite terms, but it is important as determining the imaginary part of α and thus the rate of dissipation. We may write it

$$\frac{m^2}{2a}\int_0^{\frac{1}{2}\pi} d\psi\,\frac{e^{ix\sin\psi} - 1}{\sin\psi}\{\cos(2m+2)\psi + \cos(2m-2)\psi - 2\cos 2m\psi\}, \text{...(13)}$$

where $x^2 = 4a^2\alpha^2 = 4m^2$ approximately.

Pocklington shows that the imaginary part of (13) can be expressed by means of Bessel's functions. We may take

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} d\psi\cos 2n\psi\, e^{ix\sin\psi} = J_{2n}(x) + i\,K_{2n}(x), \text{..........(14)*}$$

whence $$\int_0^{\frac{1}{2}\pi} d\psi\cos 2n\psi\,\frac{e^{ix\sin\psi} - 1}{\sin\psi} = \frac{i\pi}{2}\int_0^x \{J_{2n}(x) + i\,K_{2n}(x)\}\,dx. \text{.....(15)}$$

Accordingly, (13) may be replaced by

$$\frac{im^2\pi}{4a}\int_0^x dx\,\{J_{2m+2}(x) - 2J_{2m}(x) + J_{2m-2}(x) + i\,(K_{2m+2} - 2K_{2m} + K_{2m-2})\}. \text{ (16)}$$

Now† $$J_{2m+2} - 2J_{2m} + J_{2m-2} = 4J''_{2m},$$

so that $$\int_0^x dx\,\{J_{2m+2} - 2J_{2m} + J_{2m-2}\} = 4J'_{2m} = 2J_{2m-1} - 2J_{2m+1}. \text{.....(17)}$$

The imaginary part of (13) is thus simply

$$\frac{im^2\pi}{2a}\{J_{2m-1}(x) - J_{2m+1}(x)\}. \text{.......................(18)}$$

A corresponding theory for the K functions does not appear to have been developed.

When $m = 1$, our equation becomes

$$\left(\frac{x^2}{4} - 1\right)\log\frac{8a}{\epsilon} = -\frac{i\pi}{2}\{J_1(x) - J_3(x)\} + \frac{x^2}{3} - 2$$
$$-\int_0^{\frac{1}{2}\pi} d\psi\,\frac{\cos(x\sin\psi) - 1}{2\sin\psi}(1 - 2\cos 2\psi + \cos 4\psi), \text{......(19)}$$

* Compare *Theory of Sound*, § 302. † Gray and Mathews, *Bessel's Functions*, p. 13.

and on the right we may replace x by its first approximate value. Referring to (2) we see that the negative sign must be chosen for α and x, so that $x = -2$. The imaginary term on the right is thus

$$\frac{i\pi}{2}\{J_1(2) - J_3(2)\} = 0{\cdot}70336i.$$

For the real term Pocklington calculates 0·485, so that, L being written for $\log(8a/\epsilon)$,

$$-\alpha = \frac{1}{a}\{1 + (0{\cdot}243 + 0{\cdot}352i)/L\}. \quad\text{.....................(20)}$$

"Hence the period of the oscillation is equal to the time required for a free wave to traverse a distance equal to the circumference of the circle multiplied by $1 - 0{\cdot}243/L$, and the ratio of the amplitudes of consecutive vibrations is $1 : e^{-2{\cdot}21/L}$ or $1 - 2{\cdot}21/L$."

For the general value of m (19) is replaced by

$$(a^2\alpha^2 - m^2)\,L = \frac{im^2\pi}{2}\{J_{2m-1}(2m) - J_{2m+1}(2m)\} + R, \quad\text{.......(21)}$$

where R is a real finite number, and finally

$$-\alpha = \frac{m}{a}\left[1 + \frac{R}{2m^2L} + \frac{i\pi}{4L}\{J_{2m-1}(2m) - J_{2m+1}(2m)\}\right]. \quad\text{......(22)}$$

The ratio of the amplitudes of successive vibrations is thus

$$1 : 1 - \pi^2\{J_{2m-1}(2m) - J_{2m+1}(2m)\}/2L, \quad\text{..............(23)}$$

in which the values of $J_{2m-1}(2m) - J_{2m+1}(2m)$ can be taken from the tables (see Gray and Mathews). We have as far as m equal to 12:

m	$J_{2m-1}(2m) - J_{2m+1}(2m)$	m	$J_{2m-1}(2m) - J_{2m+1}(2m)$
1	0·448	7	0·136
2	0·298	8	0·125
3	0·232	9	0·116
4	0·194	10	0·108
5	0·169	11	0·102
6	0·150	12	0·096

It appears that the damping during a *single vibration* diminishes as m increases, viz., the greater the number of subdivisions of the circumference.

An approximate expression for the tabulated quantity when m is large may be at once derived from a formula due to Nicholson*, who shows that

* *Phil. Mag.* 1908, Vol. XVI. pp. 276, 277.

when n and z are large and nearly equal, $J_n(z)$ is related to Airy's integral. In fact,

$$J_n(z)=\frac{1}{\pi}\left(\frac{6}{z}\right)^{\frac{1}{3}}\int_0^\infty \cos\left\{w^3+(n-z)\left(\frac{6}{z}\right)^{\frac{1}{3}}w\right\}dw$$

$$=\frac{1}{\pi}\left(\frac{6}{z}\right)^{\frac{1}{3}}\left[\frac{\Gamma(\frac{1}{3})}{2\sqrt{3}}-(n-z)\left(\frac{6}{z}\right)^{\frac{1}{3}}\frac{\Gamma(\frac{2}{3})}{2\sqrt{3}}\right], \quad \ldots\ldots(24)$$

so that

$$J_{2m-1}(2m)-J_{2m+1}(2m)=\left(\frac{3}{m}\right)^{\frac{2}{3}}\frac{\Gamma(\frac{2}{3})}{\pi\sqrt{3}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(25)$$

If we apply this formula to $m=10$, we get 0·111 as compared with the tabular 0·108*.

It follows from (25) that the damping in each vibration diminishes without limit as m increases. On the other hand, the damping in a *given time* varies as $m^{\frac{1}{3}}$ and increases indefinitely, if slowly, with m.

We proceed to examine more in detail the character at a great distance of the vibration radiated from the ring. For this purpose we choose axes of x and y in the plane of the ring, and the coordinates (x, y, z) of any point may also be expressed as $r\sin\theta\cos\phi$, $r\sin\theta\sin\phi$, $r\cos\theta$. The contribution of an element $a\,d\phi'$ at ϕ' is given by (4). The direction cosines of this element are $\sin\phi'$, $-\cos\phi'$, 0; and those of the disturbance due to it are taken to be l, m, n. The direction of this disturbance is perpendicular to r and in the plane containing r and the element of arc $ad\phi'$. The first condition gives $lx+my+nz=0$, and the second gives

$$l\,.\,z\cos\phi'+m\,.\,z\sin\phi'-n\,(x\cos\phi'+y\sin\phi')=0;$$

so that

$$\frac{l}{(z^2+y^2)\sin\phi'+xy\cos\phi'}=\frac{-m}{(z^2+x^2)\cos\phi'+xy\sin\phi'}=\frac{n}{zy\cos\phi'-zx\sin\phi'}. \quad \ldots\ldots\ldots\ldots(26)$$

The sum of the squares of the denominators in (26) is

$$r^2\{z^2-(y\sin\phi'+x\cos\phi')^2\}.$$

Also in (4)

$$\sin^2\chi=1-\frac{(x\sin\phi'-y\cos\phi')^2}{r^2}=\frac{z^2+(x\cos\phi'+y\sin\phi')^2}{r^2}; \quad \ldots(27)$$

and thus

$$\left.\begin{aligned} r^2\,.\,l\ \sin\chi &= (z^2+y^2)\sin\phi'+xy\cos\phi', \\ -r^2\,.\,m\sin\chi &= (z^2+x^2)\cos\phi'+xy\sin\phi', \\ r^2\,.\,n\ \sin\chi &= zy\cos\phi'-zx\sin\phi'. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(28)$$

To these quantities the components P, Q, R due to the element $ad\phi'$ are proportional.

* $\log_{10}\Gamma(\frac{2}{3})=0{\cdot}13166$.

Before we can proceed to an integration there are two other factors to be regarded. The first relates to the intensity of the source situated at $ad\phi'$. To represent this we must introduce $\cos m\phi'$. Again, there is the question of phase. In $e^{i\alpha\rho}$ we have

$$\rho = r - a \sin\theta \cos(\phi' - \phi);$$

and in the denominator of (4) we may neglect the difference between ρ and r. Thus, as the components due to $ad\phi'$, we have

$$P = -\frac{\alpha^2 a e^{i\alpha r}}{r} d\phi' e^{-i\alpha a \sin\theta\cos(\phi'-\phi)} \cos m\phi' \frac{(z^2+y^2)\sin\phi' + xy\cos\phi'}{r^2}, \quad \ldots(29)$$

with similar expressions for Q and R corresponding to the right-hand members of (28). The integrals to be considered may be temporarily denoted by S, C, where

$$S,\ C = \int_{-\pi}^{+\pi} d\phi' \cos m\phi' e^{-i\zeta\cos(\phi'-\phi)} (\sin\phi',\ \cos\phi'), \quad \ldots\ldots\ldots(30)$$

ζ being written for $\alpha a \sin\theta$. Here

$$S = \tfrac{1}{2}\int_{-\pi}^{+\pi} d\phi' e^{-i\zeta\cos(\phi'-\phi)} \{\sin(m+1)\phi' - \sin(m-1)\phi'\},$$

and in this, if we write ψ for $\phi' - \phi$,

$$\sin(m+1)\phi' = \sin(m+1)\psi \,.\, \cos(m+1)\phi + \cos(m+1)\psi \,.\, \sin(m+1)\phi.$$

We thus find

$$S = \Theta_{m+1}\sin(m+1)\phi - \Theta_{m-1}\sin(m-1)\phi, \quad \ldots\ldots\ldots\ldots(31)$$

where
$$\Theta_n = \int_0^{\pi} d\psi \cos n\psi\, e^{-i\zeta\cos\psi}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$

In like manner,

$$C = \Theta_{m+1}\cos(m+1)\phi + \Theta_{m-1}\cos(m-1)\phi. \quad \ldots\ldots\ldots\ldots(33)$$

Now
$$\Theta_n = \int_0^{\pi} d\psi \cos n\psi \{\cos(\zeta\cos\psi) - i\sin(\zeta\cos\phi)\}.$$

When n is even, the imaginary part vanishes, and

$$\Theta_n = \frac{\pi J_n(\zeta)}{\cos\frac{1}{2}n\pi}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34)$$

On the other hand, when n is odd, the real part vanishes, and

$$\Theta_n = -\frac{i\pi J_n(\zeta)}{\sin\frac{1}{2}n\pi}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

Thus, when m is even, $m+1$ and $m-1$ are both odd and S and C are both pure imaginaries. But when m is odd, S and C are both real.

As functions of direction we may take P, Q, R to be proportional to

$$S\frac{z^2+y^2}{r^2} + C\frac{xy}{r^2}, \qquad -C\frac{z^2+x^2}{r^2} - S\frac{xy}{r^2}, \qquad C\frac{zy}{r^2} - S\frac{zx}{r^2}.$$

Whether m be odd or even, the three components are in the same phase. On the same scale the intensity of disturbance, represented by $P^2 + Q^2 + R^2$, is in terms of θ, ϕ

$$\cos^2\theta\,(S^2 + C^2) + \sin^2\theta\,(C\cos\phi + S\sin\phi)^2, \quad \text{...........(36)}$$

an expression whose sign should be changed when m is even. Introducing the values of C and S in terms of Θ from (31), (33), we find that $P^2 + Q^2 + R^2$ is proportional to

$$\cos^2\theta\,\{\Theta_{m+1}^2 + \Theta_{m-1}^2 + 2\Theta_{m+1}\Theta_{m-1}\cos 2m\phi\} + \sin^2\theta\cos^2 m\phi\,\{\Theta_{m+1} + \Theta_{m-1}\}^2. \quad \text{......(37)}$$

From this it appears that for directions lying in the plane of the ring ($\cos\theta = 0$) the radiation vanishes with $\cos m\phi$. The expression (37) may also be written

$$\Theta_{m+1}^2 + \Theta_{m-1}^2 + 2\Theta_{m+1}\Theta_{m-1}\cos 2m\phi - \tfrac{1}{2}\sin^2\theta\,(\Theta_{m+1} - \Theta_{m-1})^2\,(1 - \cos 2m\phi), \quad \text{......(38)}$$

or, in terms of J's, by (34), (35),

$$\pi^2\,[J_{m+1}^2 + J_{m-1}^2 - 2J_{m+1}J_{m-1}\cos 2m\phi - \tfrac{1}{2}\sin^2\theta\,(J_{m+1} + J_{m-1})^2\,(1 - \cos 2m\phi)], \quad \text{......(39)}$$

and this whether m be odd or even. The argument of the J's is $\alpha a \sin\theta$.

Along the axis of symmetry ($\theta = 0$) the expression (39) should be independent of ϕ. That this is so is verified when we remember that $J_n(0)$ vanishes except $n = 0$. The expression (39) thus vanishes altogether with θ unless $m = 1$, when it reduces to π^2 simply*. In the neighbourhood of the axis the intensity is of the order θ^{2m-2}.

In the plane of the ring ($\sin\theta = 1$) the general expression reduces to

$$\pi^2\,(J_{m+1} - J_{m-1})^2\cos^2 m\phi, \quad \text{or} \quad 4\pi^2 J_m'^2\cos^2 m\phi. \quad \text{.........(40)}$$

It is of interest to consider also the *mean* value of (39) reckoned over angular space. The mean with respect to ϕ is evidently

$$\pi^2\,[J_{m+1}^2 + J_{m-1}^2 + \tfrac{1}{2}\sin^2\theta\,(J_{m+1} + J_{m-1})^2]. \quad \text{...........(41)}$$

By a known formula in Bessel's functions

$$\{J_{m+1}(\zeta) + J_{m-1}(\zeta)\}^2 = \frac{4m^2}{\zeta^2}J_m^2(\zeta). \quad \text{................(42)}$$

For the present purpose

$$\zeta^2 = a^2\alpha^2\sin^2\theta = m^2\sin^2\theta\,;$$

and (41) becomes

$$\pi^2\,[J_{m+1}^2(\zeta) + J_{m-1}^2(\zeta) - 2J_m^2(\zeta)]. \quad \text{.................(43)}$$

* [*June* 20. Reciprocally, plane waves, travelling parallel to the axis of symmetry and incident upon the ring, excite none of the higher modes of vibration.]

To obtain the mean over angular space we have to multiply this by $\sin\theta d\theta$, and integrate from 0 to $\frac{1}{2}\pi$. For this purpose we require

$$\int_0^{\frac{1}{2}\pi} J_n^2(m\sin\theta)\sin\theta d\theta, \quad\ldots\ldots\ldots\ldots(44)$$

an integral which does not seem to have been evaluated.

By a known expansion* we have

$$J_0(2m\sin\theta\sin\tfrac{1}{2}\beta) = J_0^2(m\sin\theta) + 2J_1^2(m\sin\theta)\cos\beta + 2J_2^2(m\sin\theta)\cos 2\beta + \ldots\ldots,$$

whence

$$\int_0^{\frac{1}{2}\pi} J_0(2m\sin\theta\sin\tfrac{1}{2}\beta)\sin\theta d\theta$$

$$= \int_0^{\frac{1}{2}\pi} J_0^2(m\sin\theta)\sin\theta d\theta + 2\cos\beta\int_0^{\frac{1}{2}\pi} J_1^2(m\sin\theta)\sin\theta d\theta + \ldots\ldots$$

$$+ 2\cos n\beta\int_0^{\frac{1}{2}\pi} J_n^2(m\sin\theta)\sin\theta d\theta. \quad\ldots\ldots\ldots\ldots(45)$$

Now† for the integral on the left

$$\int_0^{\frac{1}{2}\pi} J_0(2m\sin\theta\sin\tfrac{1}{2}\beta)\sin\theta d\theta = \frac{\sin(2m\sin\frac{1}{2}\beta)}{2m\sin\frac{1}{2}\beta};$$

and thus

$$\int_0^{\frac{1}{2}\pi} J_n^2(m\sin\theta)\sin\theta d\theta = \frac{1}{2\pi m}\int_0^{\pi} d\beta\cos n\beta\,\frac{\sin(2m\sin\frac{1}{2}\beta)}{2m\sin\frac{1}{2}\beta}$$

$$= \frac{1}{\pi m}\int_0^{\frac{1}{2}\pi} d\psi\cos 2n\psi\,\frac{\sin(2m\sin\psi)}{\sin\psi} = \frac{1}{2m}\int_0^{2m} J_{2n}(x)\,dx, \quad\ldots\ldots(46)$$

as in (15). Thus the mean value of (43) is

$$\frac{\pi^2}{2m}\int_0^{2m} dx\,\{J_{2m+2}(x) + J_{2m-2}(x) - 2J_{2m}(x)\} = \frac{2\pi^2}{m}J_{2m}'(2m)$$

$$= \frac{\pi^2}{m}\{J_{2m-1}(2m) - J_{2m+1}(2m)\}, \quad\ldots\ldots(47)$$

as before.

In order to express fully the mean value of $P^2+Q^2+R^2$ at distance r, we have to introduce additional factors from (29). If $\alpha = -\alpha_1 - i\alpha_2$, $e^{i\alpha r} = e^{-i\alpha_1 r}e^{\alpha_2 r}$, and these factors may be taken to be $\alpha^4 a^2 e^{2\alpha_2 r}/r^2$. The occurrence of the factor $e^{2\alpha_2 r}$, where α_2 is positive, has a strange appearance; but, as Lamb has shown‡, it is to be expected in such cases as the present, where the vibrations to be found at any time at a greater distance correspond to an earlier vibration at the nucleus.

* Gray and Mathews, p. 28.

† *Enc. Brit.* "Wave Theory of Light," Equation (43), 1888; *Scientific Papers*, Vol. III. p. 98.

‡ *Proc. Math. Soc.* 1900, Vol. XXXII. p. 208.

The calculations just effected afford an independent estimate of the dissipation. The rate at which energy is propagated outwards away from the sphere of great radius r, is

$$-\frac{dE}{dt} = V \,.\, 4\pi r^2 \,.\, \frac{\alpha^4 a^2 e^{2\alpha_2 r}}{r^2} \frac{\pi^2}{m} \{J_{2m-1} - J_{2m+1}\}, \quad \text{...........(48)}$$

or, since τ (the period) $= 2\pi a/mV$, the loss of energy in one complete vibration is given by

$$-\frac{dE \,.\, \tau}{dt} = \frac{8\pi^4 \alpha^4 a^3 e^{2\alpha_2 r}}{m^2} \{J_{2m-1} - J_{2m+1}\}. \quad \text{..................(49)}$$

With this we have to compare the total energy to be found within the sphere. The occurrence of the factor $e^{2\alpha_2 r}$ is a complication from which we may emancipate ourselves by choosing r great in comparison with a, but still small enough to justify the omission of $e^{2\alpha_2 r}$, conditions which are reconcilable when ϵ is sufficiently small. The mean value of $P^2 + Q^2 + R^2$ at a small distance ρ from the circular axis is $2m^2/a^2\rho^2$. This is to be multiplied by $2\pi a \,.\, 2\pi\rho\, d\rho$, and integrated from ϵ to a value of ρ comparable with a, which need not be further specified. Thus

$$E = \frac{8m^2\pi^2}{a} \int \frac{d\rho}{\rho} = -\frac{8m^2\pi^2}{a} \log \epsilon\,; \quad \text{..................(50)}$$

and

$$-\frac{dE \,.\, \tau}{E\,dt} = \frac{\pi^2 \{J_{2m-1}(2m) - J_{2m+1}(2m)\}}{-\log \epsilon}, \quad \text{..................(51)}$$

in agreement with (23).

COLOURED PHOTOMETRY.

[*Philosophical Magazine*, Vol. XXIV. pp. 301, 302, 1912.]

In his recent paper on the Photometry of Lights of Different Colours* Mr H. Ives remarks:—"No satisfactory theory of the action of the flicker photometer can be said to exist. What does it actually measure? We may assume the existence of a 'luminosity sense' distinct from the colour sense....If, for instance, there exists a physiological process called into action both by coloured and uncoloured light, a measure of this would be a measure of a common property."

Very many years ago it occurred to me that the adjustment of the iris afforded just such a "physiological process"†. The iris contracts when the eye is exposed to a bright red or to a bright green light. There must therefore be some relative brightness of the two lights which tends *equally* to close the iris, and this may afford the measure required. The flicker adjustment is complete when the iris has no tendency to alter under the alternating illumination.

This question was brought home to me very forcibly, when in 1875 I fitted the whole area of the window of a small room with revolving sectors after the manner of Talbot. The intention was to observe, more conveniently than when the eye is at a small hole, the movements of vibrating bodies. The apparatus served this purpose well enough; but incidentally I was much struck with the remarkably disagreeable and even painful sensations experienced when at the beginning or end of operations the slits were revolving slowly so as to generate flashes at the rate of perhaps 3 or 4 per second. I soon learned in self-defence to keep my eyes closed during this phase; and I attributed the discomfort to a vain attempt on the part of the iris to adjust itself to fluctuating conditions.

* *Phil. Mag.* Vol. XXIV. p. 178.

† If my memory serves me, I have since read somewhere a similar suggestion, perhaps in Helmholtz.

It is clear, I think, that we have here a common element in variously coloured lights, such as might serve as the basis of coloured photometry. I suppose that there would be no particular difficulty in observing the movements of an iris, and I would suggest that experiments be undertaken to ascertain whether in fact the flicker match coincides with quiescence of the iris. Should this prove to be the case, the view suggested would be amply confirmed; otherwise, it would be necessary to turn to some of the other possibilities discussed by Mr Ives.

[1913. Mr H. C. Stevens (*Phil. Mag.* Vol. XXVI. p. 180, 1912), in connexion with the above suggestion, describes an experiment in which the *musculus sphincter pupillae* was paralysed with atropine, without changing "in any observable particular" the appearance of flicker. This observation may prove that an actual movement of the iris is not necessary to the sensation of flicker, but it can hardly be said that the iris has no *tendency* to alter because it is prevented from doing so by the paralysis of the muscle. There must be more than one step between the impression upon the retina which initiates a message to close the iris and the actual closing thereof. The flicker adjustment may, so far as appears, correspond to the absence of such messages.]

367.

ON SOME IRIDESCENT FILMS*.

[*Philosophical Magazine*, Vol. XXIV. pp. 751–755, 1912.]

THE experiments now to be described originated in an accidental observation. Some old lantern plates, from which the gelatine films had been cleaned off a few years before (probably with nitric acid), being required for use, were again placed in dilute nitric acid to ensure cleanliness. From these plates a gas flame burning over the dish was seen reflected with colour, of which the cause was not obvious. On examination in daylight a dry plate was observed to be iridescent, but so slightly that the fact might easily escape attention. But when the plate was under water and suitably illuminated, the brilliancy was remarkably enhanced. Upon this question of illumination almost everything depends. The window shutter of one of the rooms in my laboratory has an aperture about 4 inches square. In front of this the dish of water is placed and at the bottom of the dish a piece of dark coloured glass. In the water the plate under observation is *tilted*, so as to separate the reflexions of the sky as given by the plate and by the glass underneath. In this way a dark background is ensured. At the corners and edges of the plate the reflected light is white, then follow dark bands, and afterwards the colours which suggest reflexion from a thin plate. On this view it is necessary to suppose that the iridescent film is thinnest at the outside and thickens towards the interior, and further, that the material constituting the film has an index intermediate between those of the glass and of the water. In this way the general behaviour is readily explained, the fact that the colours are so feeble in air being attributed to the smallness of the optical difference between the film and the glass underneath. In the water there would be a better approach to equality between the reflexions at the outer and inner surfaces of the film.

From the first I formed the opinion that the films were due to the use of a silicate substratum in the original preparation, but as the history of the

* Read before the British Association at Dundee.

plates was unknown this conjecture could not be satisfactorily confirmed. No ordinary cleaning or wiping had any effect; to remove the films recourse must be had to hydrofluoric acid, or to a polishing operation. My friend Prof. T. W. Richards, after treating one with strong acids and other chemicals, pronounced it to be what chemists would call "very insoluble." The plates first encountered manifested (in the air) a brilliant glassy surface, but afterwards I found others showing in the water nearly or quite as good colours, but in the air presenting a smoky appearance.

Desirous of obtaining the colours as perfectly as possible, I endeavoured to destroy the reflexion from the back surface of the plate, which would, I supposed, dilute the colours due to the iridescent film. But a coating of black sealing-wax, or marine glue, did not do so much good as had been expected. The most efficient procedure was to grind the back of the plate, as is very easily done with carborundum. The colours seemed now to be as good as such colours can ever be, the black also being well developed. Doubtless the success was due in great measure to the special localized character of the illumination. The substitution of strong brine for water made no perceptible improvement.

At this stage I found a difficulty in understanding fully the behaviour of the unground plates. In some places the black would occasionally be good, while in others it had a washed-out appearance, a difference not easily accounted for. A difficulty had already been experienced in deciding upon which side of a plate the film was, and had been attributed to the extreme thinness of the plates. But a suspicion now arose that there were films upon *both* sides, and this was soon confirmed. The best proof was afforded by grinding away half the area upon one side of the plate and the other half of the area upon the other side. Whichever face was uppermost, the unground half witnessed the presence of a film by brilliant coloration.

Attempts to produce silicate films on new glass were for some time an almost complete failure. I used the formula given by Abney (*Instruction in Photography*, 11th edition, p. 342):

Albumen	1 part.
Water	20 parts.
Silicate of Soda solution of syrupy consistency	1 part.

But whether the plates (coated upon one side) were allowed to drain and dry in the cold, or were more quickly dried off over a spirit flame or before a fire, the resulting films washed away under the tap with the slightest friction or even with no friction at all. Occasionally, however, more adherent patches were observed, which could not so easily be cleaned off. Although it did not seem probable that the photographic film proper played any part, I tried without success a superposed coat of gelatine. In view of these failures

I could only suppose that the formation of a permanent film was the work of time, and some chemical friends were of the same opinion. Accordingly a number of plates were prepared and set aside duly labelled.

Examination at intervals proved that time acted but slowly. After six months the films seemed more stable, but nothing was obtained comparable with the old iridescent plates. It is possible that the desired result might eventually be achieved in this way, but the prospect of experimenting under such conditions is not alluring. Luckily an accidental observation came to my aid. In order to prevent the precipitation of lime in the observing dish a few drops of nitric acid were sometimes added to the water, and I fancied that films tested in this acidified water showed an advantage. A special experiment confirmed the idea. Two plates, coated similarly with silicate and dried a few hours before, were immersed, one in ordinary tap water, the other in the same water moderately acidified with nitric acid. After some 24 hours' soaking the first film washed off easily, but the second had much greater fixity. There was now no difficulty in preparing films capable of showing as good colours as those of the old plates. The best procedure seems to be to dry off the plates before a fire after coating with recently filtered silicate solution. In order to obtain the most suitable thickness, it is necessary to accommodate the rapidity of drying to the strength of the solution. If heat is not employed the strength of the above given solution may be doubled. When dry the plates may be immersed for some hours in somewhat diluted nitric acid. They are then fit for optical examination, but are best not rubbed at this stage. If the colours are suitable the plates may now be washed and allowed to dry. The full development of the colour effects requires that the back of the plates be treated. In my experience grinding gives the best results when the lighting is favourable, but an opaque varnish may also be used with good effect. The comparative failure of such a treatment of the old plates was due to the existence of films upon both sides. A sufficiently opaque glass, *e.g.* stained with cobalt or copper, may also be employed. After the films have stood some time subsequently to the treatment with acid, they may be rubbed vigorously with a cloth even while wet, but one or two, which probably had been rubbed prematurely, showed scratches.

The surfaces of the new films are not quite as glassy as the best of the old ones, not so inconspicuous in the air, but there is, I suppose, no doubt that they are all composed of silica. But I am puzzled to understand how the old plates were manipulated. The films cover both sides without interruption, and are thinner at all the four corners than in the interior.

The extraordinary development of the colours in water as compared with what can be seen in air led me to examine in the same way other thin films deposited on glass. A thin coat of albumen (without silicate) is inconspicuous

in air. As in photography it may be rendered insoluble by nitrate of silver acidified with acetic acid, and then exhibits good colours when examined under water with favourable illumination. Filtered gelatine, with which a little bichromate has been mixed beforehand, may also be employed. In this case the dry film should be well exposed to light before washing. Ready-made varnishes also answer well, provided they are capable of withstanding the action of water, at least for a time. I have used amber in chloroform, a "crystal" (benzole) varnish such as is, or was, used by photographers, and bitumen dissolved in benzole. The last is soon disintegrated under water, but the crystal varnish gives very good films. The varnish as sold may probably require dilution in order that the film may be thin enough.

Another varnish which gives interesting results is celluloid in pear-oil. All these films show little in air, but display beautiful colours in water when the reflexion from the back of the glass is got rid of as already described. The advantage from the water depends, of course, upon its mitigating the inequality of the reflexion from the two sides of the film by diminishing the front reflexion. A similar result may be arrived at by another road if we can increase the back reflexion, with the further advantage of enhanced illumination. For this purpose we may use silvering. A glass is coated with a very thin silver film and then with celluloid varnish of suitable consistency. Magnificent colours are then seen without the aid of water, and the only difficulty is to hit off the right thickness for the silver. Other methods of obtaining similar displays are described in Wood's *Physical Optics* (Macmillan, 1905, p. 142).

368.

BREATH FIGURES*.

[*Nature*, Vol. xc, pp. 436, 437, 1912.]

At intervals during the past year I have tried a good many experiments in the hope of throwing further light upon the origin of these figures, especially those due to the passage of a small blow-pipe flame, or of hot sulphuric acid, across the surface of a glass plate on which, before treatment, the breath deposits evenly. The even deposit consists of a multitude of small lenses easily seen with a hand magnifier. In the track of the flame or sulphuric acid the lenses are larger, often passing into flat masses which on evaporation show the usual colours of thin plates. When the glass is seen against a dark ground, and is so held that regularly reflected light does not reach the eye, the general surface shows bright, while the track of the flame or acid is by comparison dark or black. It will be convenient thus to speak of the deposit as bright or dark—descriptive words implying no doubtful hypothesis. The question is what difference in the glass surface determines the two kinds of deposit.

In Aitken's view (*Proc. Ed. Soc.* p. 94, 1893; *Nature*, June 15, 1911), the flame acts by the deposit of numerous fine particles constituting nuclei of aqueous condensation, and in like manner he attributes the effect of sulphuric (or hydrofluoric) acid to a water-attracting residue remaining in spite of washing. On the other hand, I was disposed to refer the dark deposit to a greater degree of freedom from grease or other water-repelling contamination (*Nature*, May 25, 1911), supposing that a clean surface of glass would everywhere attract moisture. It will be seen that the two views are sharply contrasted.

My first experiments were directed to improving the washing after hot sulphuric or hydrofluoric acid. It soon appeared that rinsing and soaking prolonged over twenty-four hours failed to abolish the dark track; but probably Mr Aitken would not regard this as at all conclusive. It was more to the point that dilute sulphuric acid (1/10) left no track, even after perfunctory washing. Rather to my surprise, I found that even strong

* See p. 26 of this volume.

sulphuric acid fails if employed cold. A few drops were poured upon a glass (¼-plate photographic from which the film had been removed), and caused to form an elongated pool, say, half an inch wide. After standing level for about five minutes—longer than the time required for the treatment with hot acid—the plate was rapidly washed under the tap, soaked for a few minutes, and finally rinsed with distilled water, and dried over a spirit lamp. Examined when cold by breathing, the plate showed, indeed, the form of the pool, but mainly by the darkness of the *edge*. The interior was, perhaps, not quite indistinguishable from the ground on which the acid had not acted, but there was no approach to darkness. This experiment may, I suppose, be taken to prove that the action of the hot acid is not attributable to a residue remaining after the washing.

I have not found any other treatment which will produce a dark track without the aid of heat. Chromic acid, *aqua regia*, and strong potash are alike ineffective. These reagents do undoubtedly exercise a cleansing action, so that the result is not entirely in favour of the grease theory as ordinarily understood.

My son, Hon. R. J. Strutt, tried for me an experiment in which part of an ordinarily cleaned glass was exposed for three hours to a stream of strongly ozonised oxygen, the remainder being protected. On examination with the breath, the difference between the protected and unprotected parts was scarcely visible.

It has been mentioned that the edges of pools of strong cold sulphuric acid and of many other reagents impress themselves, even when there is little or no effect in the interior. To exhibit this action at its best it is well to employ a minimum of liquid; otherwise a *creeping of the edge* during the time of contact may somewhat obscure it. The experiment succeeds about equally well even when distilled water from a wash bottle is substituted for powerful reagents. On the grease theory the effect may be attributed to the cleansing action of a pure free surface, but other interpretations probably could be suggested.

Very dark deposits, showing under suitable illumination the colours of thin plates, may be obtained on freshly-blown bulbs of soft glass. It is convenient to fill the interior with water, to which a little ink may be added. From this observation no particular conclusion can be deduced, since the surface, though doubtless very clean, has been exposed to the blow-pipe flame. In my former communication, I mentioned that no satisfactory result was obtained when a glass plate was strongly heated *on the back* by a long Bunsen burner; but I am now able to bring forward a more successful experiment.

A test-tube of thin glass, about ¾ inch in diameter, was cleaned internally until it gave an even bright deposit. The breath is introduced through

a tube of smaller diameter, previously warmed slightly with the hand. The closed end of the test-tube was then heated in a gas flame urged with a foot blow-pipe until there were signs of incipient softening. After cooling, the breath deposit showed interesting features, best brought out by transmitted light under a magnifier. The greater part of the length showed, as before, the usual fine dew. As the closed end was approached the drops became gradually larger, until at about an inch from the end they disappeared, leaving the glass covered with a nearly uniform film. One advantage of the tube is that evaporation of dew, once formed, is slow, unless promoted by suction through the mouth tube. As the film evaporated, the colours of thin plates were seen by reflected light. Since it is certain that the flame had no access to the internal surface, it seems proved that dark deposits can be obtained on surfaces treated by heat alone.

In some respects a tube of thin glass, open at *both* ends, is more convenient than the test-tube. It is easier to clean, and no auxiliary tube is required to introduce or abstract moisture. I have used one of $\frac{3}{10}$ in. diameter. Heated locally over a simple spirit flame to a point *short of softening*, it exhibited similar effects. This easy experiment may be recommended to anyone interested in the subject.

One of the things that I have always felt as a difficulty is the comparative permanence of the dark tracts. On flat plates they may survive in some degree rubbing by the finger, with subsequent rinsing and wiping. Practically the easiest way to bring a plate back to its original condition is to rub it with soapy water. But even this does not fully succeed with the test-tube, probably on account of the less effective rubbing and wiping near the closed end. But what exactly is involved in rubbing and wiping? I ventured to suggest before that possibly grease may penetrate the glass somewhat. From such a situation it might not easily be removed, or, on the other hand, introduced.

There is another form of experiment from which I had hoped to reap decisive results. The interior of a mass of glass cannot be supposed to be greasy, so that a surface freshly obtained by fracture should be clean, and give the dark deposit. One difficulty is that the character of the deposit on the irregular surface is not so easily judged. My first trial on a piece of plate glass [illegible] in. thick, broken into two pieces with a hammer, gave anomalous results. On part of each new surface the breath was deposited in thin lamina capable of showing colours, but on another part the water masses were decidedly smaller, and the deposit could scarcely be classified as black. The black and less black parts of the two surfaces were those which had been contiguous before fracture. That there should be a well-marked difference in this respect between parts both inside a rather small piece of glass is very surprising. I have not again met with this anomaly; but

further trials on thick glass have revealed deposits which may be considered dark, though I was not always satisfied that they were so dark as those obtained on flat surfaces with the blow-pipe or hot sulphuric acid. Similar experiments with similar results may be made upon the edges of ordinary glass plates (such as are used in photography), cut with a diamond. The breath deposit is best held pretty close to a candle-flame, and is examined with a magnifier.

In conclusion, I may refer to two other related matters in which my experience differs from that of Mr Aitken. He mentions that with an alcohol flame he "could only succeed in getting very slight indications of any action." I do not at all understand this, as I have nearly always used an alcohol flame (with a mouth blow-pipe) and got black deposits. Thinking that perhaps the alcohol which I generally use was contaminated, I replaced it by pure alcohol, but without any perceptible difference in the results.

Again, I had instanced the visibility of a gas flame through a dewed plate as proving that part of the surface was uncovered. I have improved the experiment by using a curved tube through which to blow upon a glass plate already in position between the flame and the eye. I have not been able to find that the flame becomes invisible (with a well-defined outline) at any stage of the deposition of dew. Mr Aitken mentions results pointing in the opposite direction. Doubtless, the highly localized light of the flame is favourable.

[1913. Mr Aitken returned to the subject in a further communication to *Nature*, Vol. xc. p. 619, 1912, to which the reader should refer.]

369.

REMARKS CONCERNING FOURIER'S THEOREM AS APPLIED TO PHYSICAL PROBLEMS.

[*Philosophical Magazine*, Vol. XXIV. pp. 864—869, 1912.]

FOURIER'S theorem is of great importance in mathematical physics, but difficulties sometimes arise in practical applications which seem to have their origin in the aim at too great a precision. For example, in a series of observations extending over time we may be interested in what occurs during seconds or years, but we are not concerned with and have no materials for a remote antiquity or a distant future; and yet these remote times determine whether or not a period precisely defined shall be present. On the other hand, there may be no clearly marked limits of time indicated by the circumstances of the case, such as would suggest the other form of Fourier's theorem where everything is ultimately periodic. Neither of the usual forms of the theorem is exactly suitable. Some method of taking off the edge, as it were, appears to be called for.

The considerations which follow, arising out of a physical problem, have cleared up my own ideas, and they may perhaps be useful to other physicists.

A train of waves of length λ, represented by

$$\psi = e^{2\pi i(ct+x)/\lambda}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

advances with velocity c in the negative direction. If the medium is absolutely uniform, it is propagated without disturbance; but if the medium is subject to small variations, a reflexion in general ensues as the waves pass any place x. Such reflexion reacts upon the original waves; but if we suppose the variations of the medium to be extremely small, we may neglect the reaction and calculate the aggregate reflexion as if the primary waves were undisturbed. The partial reflexion which takes place at x is represented by

$$d\psi = e^{2\pi i(ct-x)/\lambda}\,\phi(x)\,dx\,.\,e^{4\pi ix/\lambda}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

in which the first factor expresses total reflexion supposed to originate at $x=0$, $\phi(x)\,dx$ expresses the actual reflecting power at x, and the last factor gives the alteration of phase incurred in traversing the distance $2x$. The aggregate reflexion follows on integration with respect to x; with omission of the first factor it may be taken to be

$$C+iS, \qquad\qquad (3)$$

where
$$C=\int_{-\infty}^{+\infty}\phi(v)\cos uv\,dv, \quad S=\int_{-\infty}^{+\infty}\phi(v)\sin uv\,dv, \qquad (4)$$

with $u=4\pi/\lambda$. When ϕ is given, the reflexion is thus determined by (3). It is, of course, a function of λ or u.

In the converse problem we regard (3)—the reflexion—as given for all values of u and we seek thence to determine the form of ϕ as a function of x. By Fourier's theorem we have at once

$$\phi(x)=\frac{1}{\pi}\int_0^\infty du\,\{C\cos ux+S\sin ux\}. \qquad (5)$$

It will be seen that we require to know C and S separately. A knowledge of the *intensity* merely, viz. C^2+S^2, does not suffice.

Although the general theory, above sketched, is simple enough, questions arise as soon as we try to introduce the approximations necessary in practice. For example, in the optical application we could find by observation the values of C and S for a finite range only of u, limited indeed in eye observations to less than an octave. If we limit the integration in (5) to correspond with actual knowledge of C and S, the integral may not go far towards determining ϕ. It may happen, however, that we have some independent knowledge of the form of ϕ. For example, we may know that the medium is composed of strata each uniform in itself, so that within each ϕ vanishes. Further, we may know that there are only two kinds of strata, occurring alternately. The value of $\int\phi\,dx$ at each transition is then numerically the same but affected with signs alternately opposite. This is the case of chlorate of potash crystals in which occur repeated twinnings*. Information of this kind may supplement the deficiency of (5) taken by itself. If it be for high values only of u that C and S are not known, the curve for ϕ first obtained may be subjected to any alteration which leaves $\int\phi\,dx$, taken over any small range, undisturbed, a consideration which assists materially where ϕ is known to be discontinuous.

If observation indicates a large C or S for any particular value of u, we infer of course from (5) a correspondingly important periodic term in ϕ. If the large value of C or S is limited to a very small range of u, the periodicity of ϕ extends to a large range of x; otherwise the interference of

* *Phil. Mag.* Vol. XXVI. p. 256 (1888); *Scientific Papers*, Vol. III. p. 204.

components with somewhat different values of u may limit the periodicity to a comparatively small range. Conversely, a prolonged periodicity is associated with an approach to discontinuity in the values of C or S.

The complete curve representing $\phi(x)$ will in general include features of various lengths reckoned along x, and a feature of any particular length is associated with values of u grouped round a corresponding centre. For some purposes we may wish to *smooth* the curve by eliminating small features. One way of effecting this is to substitute everywhere for $\phi(x)$ the mean of the values of $\phi(x)$ in the neighbourhood of x, viz.

$$\frac{1}{2a}\int_{x-a}^{x+a}\phi(x)\,dx, \quad \text{.............................(6)}$$

the range ($2a$) of integration being chosen suitably. With use of (5) we find for (6)

$$\frac{1}{2a}\int_{x-a}^{x+a}\phi(x)\,dx = \frac{1}{\pi}\int_0^\infty du\,\frac{\sin ua}{ua}\{C\cos ux + S\sin ux\}, \quad \text{.......(7)}$$

differing from the right-hand member of (5) merely by the introduction of the factor $\sin ua \div ua$. The effect of this factor under the integral sign is to diminish the importance of values of u which exceed π/a and gradually to annul the influence of still larger values. If we are content to speak very roughly, we may say that the process of averaging on the left is equivalent to the omission in Fourier's integral of the values of u which exceed $\pi/2a$.

We may imagine the process of averaging to be repeated once or more times upon (6). At each step a new factor $\sin ua \div ua$ is introduced under the integral sign. After a number of such operations the integral becomes practically independent of all values of u for which ua is not small.

In (6) the average is taken in the simplest way with respect to x, so that every part of the range $2a$ contributes equally (fig. 1). Other and perhaps

Fig. 1. Fig. 2. Fig. 3.

better methods of smoothing may be proposed in which a preponderance is given to the central parts. For example we may take (fig. 2)

$$\frac{1}{a^2}\int_0^a (a-\xi)\{\phi(x+\xi)+\phi(x-\xi)\}\,d\xi. \quad \text{.................(8)}$$

From (5) we find that (8) is equivalent to

$$\frac{2}{\pi}\int_0^\infty du\,\frac{1-\cos ua}{u^2a^2}\{C\cos ux + S\sin ux\}, \quad \text{................(9)}$$

reducing to (5) again when a is made infinitely small. In comparison with (7) the higher values of ua are eliminated more rapidly. Other kinds of averaging over a finite range may be proposed. On the same lines as above the formula next in order is (fig. 3)

$$\frac{3}{4a}\int_{-a}^{+a}\left(1-\frac{\xi^2}{a^2}\right)\phi(x+\xi)\,d\xi$$
$$=\frac{1}{\pi}\int_0^\infty du\,\frac{\sin au - au\cos au}{\frac{1}{3}a^3u^3}\,\{C\cos ux + S\sin ux\}\,dx. \quad\ldots(10)$$

In the above processes for smoothing the curve representing $\phi(x)$, ordinates which lie at distances exceeding a from the point under consideration are without influence. This may or may not be an advantage. A formula in which the integration extends to infinity is

$$\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{+\infty}\phi(x+\xi)\,e^{-\xi^2/a^2}\,d\xi = \frac{1}{\pi}\int_0^\infty du\,e^{-u^2a^2/4}\,\{C\cos ux + S\sin ux\}. \quad\ldots(11)$$

In this case the values of ua which exceed 2 make contributions to the integral whose importance very rapidly diminishes.

The intention of the operation of smoothing is to remove from the curve features whose length is small. For some purposes we may desire on the contrary to eliminate features of *great* length, as for example in considering the record of an instrument whose zero is liable to slow variation from some extraneous cause. In this case (to take the simplest formula) we may subtract from $\phi(x)$—the uncorrected record—the average over a length b relatively large, so obtaining

$$\phi(x)-\frac{1}{2b}\int_{x-b}^{x+b}\phi(x)\,dx = \frac{1}{\pi}\int_0^\infty du\left\{1-\frac{\sin ub}{ub}\right\}\{C\cos ux + S\sin ux\}. \quad\ldots(12)$$

Here, if ub is much less than π, the corresponding part of the range of integration is approximately cancelled and features of great length are eliminated.

There are cases where this operation and that of smoothing may be combined advantageously. Thus if we take

$$\frac{1}{2a}\int_{x-a}^{x+a}\phi(x)\,dx - \frac{1}{2b}\int_{x-b}^{x+b}\phi(x)\,dx$$
$$=\frac{1}{\pi}\int_0^\infty du\left\{\frac{\sin ua}{ua}-\frac{\sin ub}{ub}\right\}\{C\cos ux + S\sin ux\}, \quad\ldots(13)$$

we eliminate at the same time the features whose length is small compared with a and those whose length is large compared with b. The same method may be applied to the other formulæ (9), (10), (11).

A related question is one proposed by Stokes*, to which it would be interesting to have had Stokes' own answer. What is in common and what

* Smith's Prize Examination, Feb. 1, 1882; *Math. and Phys. Papers*, Vol. v. p. 367.

is the difference between C and S in the two cases (i) where $\phi(x)$ fluctuates between $-\infty$ and $+\infty$ and (ii) where the fluctuations are nearly the same as in (i) between finite limits $\pm a$ but outside those limits tends to zero? When x is numerically great, $\cos ux$ and $\sin ux$ fluctuate rapidly with u; and inspection of (5) shows that $\phi(x)$ is then small, unless C or S are themselves rapidly variable as functions of u. Case (i) therefore involves an approach to discontinuity in the forms of C or S. If we eliminate these discontinuities, or rapid variations, by a smoothing process, we shall annul $\phi(x)$ at great distances and at the same time retain the former values near the origin. The smoothing may be effected (as before) by taking

$$\frac{1}{2a}\int_{u-a}^{u+a} C\,du, \quad \frac{1}{2a}\int_{u-a}^{u+a} S\,du$$

in place of C and S simply. C then becomes

$$\int_{-\infty}^{+\infty} dv\,\phi(v)\cos uv\,\frac{\sin av}{av},$$

$\phi(v)$ being replaced by $\phi(v)\sin av \div av$. The effect of the added factor disappears when av is small, but when av is large, it tends to annul the corresponding part of the integral. The new form for $\phi(x)$ is thus the same as the old one near the origin but tends to vanish at great distances on either side. Case (ii) is thus deducible from case (i) by the application of a smoothing process to C and S, whereby fluctuations of small length are removed.

We may sum up by saying that a smoothing of $\phi(x)$ annuls C and S for large values of u, while a smoothing of C and S (as functions of u) annuls $\phi(x)$ for values of x which are numerically great.

370.

SUR LA RÉSISTANCE DES SPHÈRES DANS L'AIR EN MOUVEMENT.

[*Comptes Rendus*, t. CLVI. p. 109, 1913.]

DANS les *Comptes rendus* du 30 décembre 1912, M. Eiffel donne des résultats très intéressants pour la résistance rencontrée, à vitesse variable, par trois sphères de 16·2, 24·4 et 33 cm. de diamètre. Dans la première figure, ces résultats sont exprimés par les valeurs d'un coefficient K, égal à R/SV^2, où R est la résistance totale, S la surface diamétrale et V la vitesse. En chaque cas, il y a une *vitesse critique*, et M. Eiffel fait remarquer que la *loi de similitude* n'est pas toujours vraie ; en effet, les trois sphères donnent des vitesses critiques tout à fait différentes.

D'après la loi de similitude dynamique, précisée par Stokes* et Reynolds pour les liquides visqueux, K est une fonction d'une *seule* variable ν/VL, où ν est la *viscosité cinématique*, constante pour un liquide donné, et L est la dimension linéaire, proportionnelle à $S^{\frac{1}{2}}$. Ainsi les vitesses critiques ne doivent pas être les mêmes dans les trois cas, mais inversement proportionnelles à L. En vérité, si nous changeons l'échelle des vitesses suivant cette loi, nous trouvons les courbes de M. Eiffel presque identiques, au moins que ces vitesses ne sont pas très petites.

Je ne sais si les écarts résiduels sont réels ou non. La théorie simple admet que les sphères sont polies, sinon que les inégalités sont proportionnelles aux diamètres, que la compressibilité de l'air est négligeable et que la viscosité cinématique est absolument constante. Si les résultats de l'expérience ne sont pas complètement d'accord avec la théorie, on devra examiner ces hypothèses de plus près.

J'ai traité d'autre part et plus en détail de la question dont il s'agit ici†.

* [*Camb. Trans.* 1850; *Math. and Phys. Papers*, Vol. III. p. 17.]

† Voir *Scientific Papers*, t. V. 1910, pp. 532—534.

371.

THE EFFECT OF JUNCTIONS ON THE PROPAGATION OF ELECTRIC WAVES ALONG CONDUCTORS.

[*Proceedings of the Royal Society*, A, Vol. LXXXVIII. pp. 103–110, 1913.]

SOME interesting problems in electric wave propagation are suggested by an experiment of Hertz*. In its original form waves of the simplest kind travel in the positive direction (fig. 1), outside an infinitely thin conducting cylindrical shell, AA, which comes to an end, say, at the plane $z = 0$. Co-axial with the cylinder a rod or wire BB (of less diameter) extends to infinity in both directions. The conductors being supposed perfect, it is required to determine the waves propagated onwards beyond the cylinder on the positive side of z, as well as those reflected back outside the cylinder and in the annular space between the cylinder and the rod.

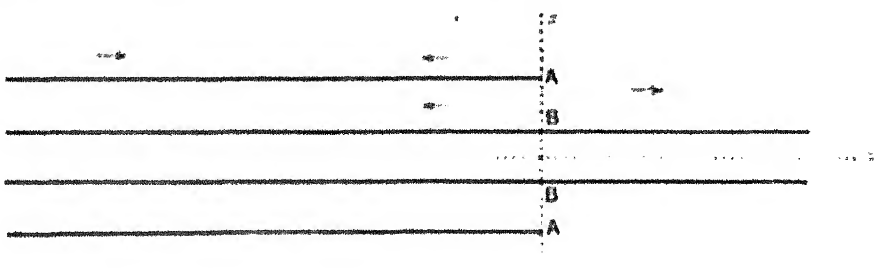


Fig. 1

So stated, the problem, even if mathematically definite, is probably intractable; but if we modify it by introducing an external co-axial conducting sheath CC (fig. 2), extending to infinity in both directions, and if we further suppose that the diameter of this sheath is small in comparison with the wave-length (λ) of the vibrations, we shall bring it within the scope of approximate methods. It is under this limitation that I propose here to

* "Ueber die Fortleitung electrischer Wellen durch Drähte," *Wied. Ann.* 1889, Vol. XXXVII. p. 395.

consider the present and a few analogous problems. Some considerations of a more general character are prefixed.

If P, Q, R be components of electromotive intensity, a, b, c those of magnetisation, Maxwell's general circuital relations* for the dielectric give

$$\frac{da}{dt}=\frac{dQ}{dz}-\frac{dR}{dy}, \quad \text{.............................(1)}$$

and two similar equations, and

$$\frac{dP}{dt}=V^2\left(\frac{dc}{dy}-\frac{db}{dz}\right), \quad \text{.............................(2)}$$

also with two similar equations, V being the velocity of propagation. From (1) and (2) we may derive

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0, \qquad \frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}=0; \quad \text{...............(3)}$$

and, further, that $$\left(\frac{d^2}{dt^2}-V^2\nabla^2\right)(P,\,Q,\,R,\,a,\,b,\,c)=0, \quad \text{..................(4)}$$

where $$\nabla^2=d^2/dx^2+d^2/dy^2+d^2/dz^2. \quad \text{......................(5)}$$

At any point upon the surface of a conductor, regarded as perfect, the condition to be satisfied is that the vector (P, Q, R) be there *normal*. In what follows we shall have to deal only with simple vibrations in which all the quantities are proportional to e^{ipt}, so that d/dt may be replaced by ip.

It may be convenient to commence with some cases where the waves are in two dimensions (x, z) only, supposing that a, c, Q vanish, while b, P, R are independent of y. From (1) and (2) we have

$$P\frac{db}{dx}+R\frac{db}{dz}=0.$$

At the surface of a conductor P, Q are proportional to the direction cosines of the normal (n); so that the surface condition may be expressed simply by

$$\frac{db}{dn}=0, \quad \text{....................................(6)}$$

which, with $$\left(\frac{d^2}{dx^2}+\frac{d^2}{dz^2}+k^2\right)b=0, \quad \text{.........................(7)}$$

suffices to determine b. In (7) $k=p/V$. It will be seen that equations (6), (7) are identical with those which apply in two dimensions to aërial vibrations executed in spaces bounded by fixed walls, b then denoting velocity-potential. When b is known, the remaining functions follow at once.

* *Phil. Trans.* 1868; Maxwell's *Scientific Papers*, Vol. II. p. 128.

It may be remarked by the way that the above analogy throws light upon the question under what circumstances electric waves are *guided* by conductors. Some high authorities, it would seem, regard such guidance as ensuing in all cases as a consequence of the boundary condition fixing the direction of the electric force. But in Acoustics, though a similar condition holds good, there is no guidance of aerial waves round convex surfaces, and it follows that there is none in the two-dimensional electric vibrations under consideration. Near the concave surface of walls there is in both cases a whispering gallery effect*. The peculiar guidance of electric waves by wires depends upon the conductor being encircled by the magnetic force. No such circulation, for example, could ensue from the incidence of plane waves upon a wire which lies entirely in the plane containing the direction of propagation and that of the magnetic force.

Our first special application is to the extreme form of Hertz's problem (as modified) which occurs when all the radii of the cylindrical surfaces concerned become infinite, while the *differences* CA, AB remain finite and indeed small in comparison with λ. In fig. 2, A, B, C then represent

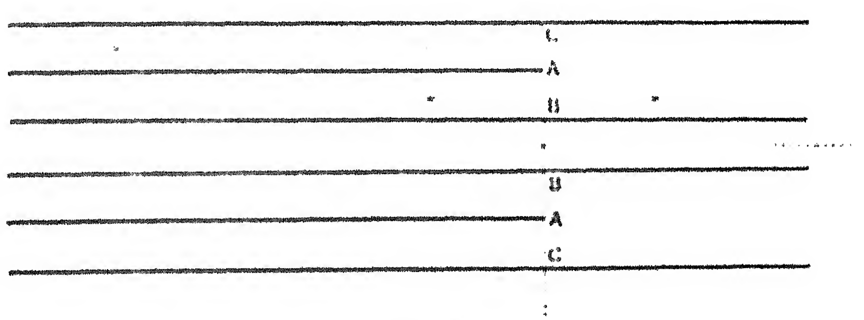


Fig. 2.

planes perpendicular to the plane of the paper and the problem is in two dimensions. The two halves, corresponding to *plus* and *minus* values of x, are isolated, and we need only consider one of them. Availing ourselves of the acoustical analogy, we may at once transfer the solution given (after Poisson) in *Theory of Sound*, § 264. If the incident wave in CA be represented by f_1, and that therein reflected by F, while the waves propagated along CB, AB be denoted by f_2, f_3, we have

$$f_2 = f_3 = \frac{2CA}{CB + AB + CA} f_1 = \frac{CA}{CB} f_1 \quad \text{..............(8)}$$

and

$$F = \frac{AB}{CB} f_1 \quad \text{....................................(9)}$$

* *Phil. Mag.* 1910, Vol. xx. p. 1001; *Scientific Papers*, Vol. v. p. 617.

The wave in AB is to be regarded as propagated onwards round the corner at A rather than as reflected. As was to be anticipated, the reflected wave F' is smaller, the smaller is AB. It will be understood that the validity of these results depends upon the assumption that the region round A through which the waves are irregular has dimensions which are negligible in comparison with λ.

An even simpler example is sketched in fig. 3, where for the present the

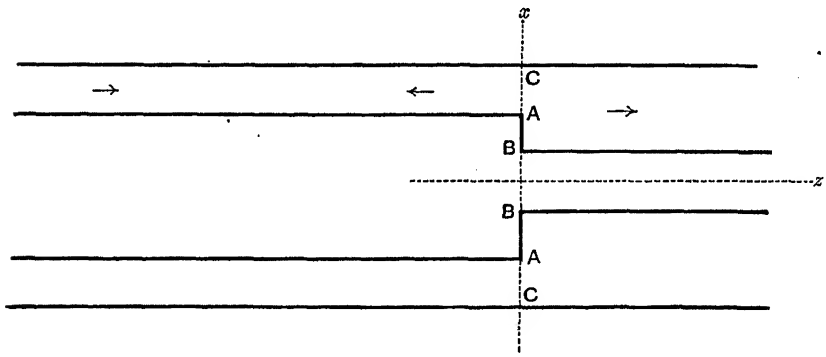


Fig. 3.

various lines represent planes or cylindrical surfaces perpendicular to the paper. One bounding plane C is unbroken. The other boundary consists mainly of two planes with a transition at AB, which, however, may be of any form so long as it is effected within a distance much less than λ. With a notation similar to that used before, f_{CA} may denote the incident positive wave and F the reflected wave, while that propagated onwards in CB is f_{CB}. We obtain in like manner

$$f'_{CB} = \frac{2CA}{CB + CA} f'_{CA}; \quad \text{.........................(10)}$$

$$F' = \frac{AB}{CB + CA} f'_{CA}. \quad \text{.........................(11)}$$

When AB vanishes we have, of course, $f'_{CB} = f'_{CA}$, $F' = 0$. A little later we shall consider the problem of fig. 3 when the various surfaces are of revolution round the axis of z.

Leaving the two-dimensional examples, we find that the same general method is applicable, always under the condition that the region occupied by irregular waves has dimensions which are small in comparison with λ. Within this region a simplified form of the general equations avails, and thus the difficulty is turned.

An increase in λ means a decrease in p. When this goes far enough, it justifies the omission of d/dt in equations (1), (2), (3), (4). Thus P, Q, R become the derivatives of a simple potential function ϕ, which itself satisfies

$\nabla^2\phi = 0$, that is, the electric forces obey the laws of electrostatics. Similarly a, b, c are derivatives of another function ψ satisfying the same equation. The only difference is that ψ may be multivalued. The magnetism is that due to steady electric currents. If several wires meet in a point, the total current is *zero*. This expresses itself in terms of a, b, c as a relation between the "circulations." The method then consists in forming the solutions which apply to the parts at a distance on the two sides from the region of irregularity, and in accommodating them to one another by the conditions which hold good at the margins of this region in virtue of the fact that it is small.

In the application to the problem of fig. 3 we will suppose that the conductors are of revolution round z, though this limitation is not really imposed by the method itself. The problem of the regular waves (whatever may be form of section) was considered in a former paper*. All the dependent variables expressing the electric conditions being proportional to $e^{i(pt\pm kz)}$, d^2/dt^2 in (4) compensates V^2d^2/dz^2, so that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)(P, Q, R, a, b, c) = 0; \quad \text{..................(12)}$$

also R and c vanish. In the present case we have for the negative side, where there is both a direct and a reflected wave,

$$P, Q, R = e^{ipt}(H_1e^{-ikz} + K_1e^{ikz})\left(\frac{d}{dx}, \frac{d}{dy}, 0\right)\log r, \quad \text{........(13)}$$

where r is the distance of any point from the axis of symmetry, and H_1, K_1 are arbitrary constants. Corresponding to (13),

$$V(a, b, c) = e^{ipt}(-H_1e^{-ikz} + K_1e^{ikz})\left(\frac{d}{dy}, -\frac{d}{dx}, 0\right)\log r. \quad \text{.....(14)}$$

In the region of regular waves on the positive side there is supposed to be no wave propagated in the negative direction. Here accordingly

$$P, Q, R = H_2e^{i(pt-kz)}\left(\frac{d}{dx}, \frac{d}{dy}, 0\right)\log r, \quad \text{.............(15)}$$

$$V(a, b, c) = H_2e^{i(pt-kz)}\left(-\frac{d}{dy}, \frac{d}{dx}, 0\right)\log r, \quad \text{..........(16)}$$

H_2 being another constant. We have now to determine the relations between the constants H_1, K_1, H_2, hitherto arbitrary, in terms of the remaining data.

For this purpose consider cross-sections on the two sides both near the origin and yet within the regions of regular waves. The electric force as expressed in (13), (15) is purely radial. On the positive side its integral

* *Phil. Mag.* 1897, Vol. XLIV. p. 199; *Scientific Papers*, Vol. IV. p. 327.

between r_2 the radius of the inner and r' that of the outer conductor is, with omission of e^{ipt},

$$H_2 e^{-ikz} \log (r'/r_2),$$

z having the value proper to the section. On the negative side the corresponding integral is

$$(H_1 e^{-ikz} + K_1 e^{ikz}) \log (r'/r_1),$$

r_1 being the radius of the inner conductor at that place. But when we consider the intermediate region, where electrostatical laws prevail, we recognize that these two integrals must be equal; and further that the exponentials may be identified with unity. Accordingly, the first relation is

$$(H_1 + K_1) \log (r'/r_1) = H_2 \log (r'/r_2). \quad \text{.....................(17)}$$

In like manner the magnetic force in (14), (16) is purely circumferential. And the circulations at the two sections are as $H_1 - K_1$ and H_2. But since these circulations, representing electric currents which may be treated as steady, are equal, we have as the second relation—

$$H_1 - K_1 = H_2. \quad \text{..............................(18)}$$

The two relations (17), (18) determine the wave propagated onwards H_2 and that reflected K_1 in terms of the incident wave H_1. If $r_2 = r_1$, we have of course, $H_2 = H_1$, $K_1 = 0$.

If we suppose r_1, r_2, r' all great and nearly equal and expand the logarithms, we fall back on the solution for the two-dimensional case already given.

In the above the radius of the outer sheath is supposed uniform throughout. If in the neighbourhood of the origin the radius of the sheath changes from r_1' to r_2', while (as before) that of the inner conductor changes from r_1 to r_2, we have instead of (17),

$$(H_1 + K_1) \log (r_1'/r_1) = H_2 \log (r_2'/r_2), \quad \text{................(19)}$$

while (18) remains undisturbed.

In (19) the logarithmic functions are proportional to the reciprocals of the electric capacities of the system on the two sides, reckoned in each case per unit of length. From the general theory given in the paper referred to we may infer that this substitution suffices to liberate us from the restriction to symmetry round the axis hitherto imposed. The more general functions which then replace $\log r$ on the two sides must be chosen with such coefficients as will make the circulations of magnetic force equal. The generalization here indicated applies equally in the other problems of this paper.

In Hertz's problem, fig. 2, the method is similar. In the region of regular waves on the left in CA we may retain (13), (14), and for the regular waves on the right in CB we retain (15), (16). But now in addition for the regular waves on the left in AB, we have

$$P, Q, R = K_3 e^{ipt+ikz}\left(\frac{d}{dx}, \frac{d}{dy}, 0\right)\log r, \quad \text{..............(20)}$$

$$V(a, b, c) = K_3 e^{ipt+ikz}\left(\frac{d}{dy}, -\frac{d}{dx}, 0\right)\log r. \quad \text{...........(21)}$$

Three conditions are now required to determine K_1, H_2, K_3 in terms of H_1. We shall denote the radii taken in order, viz. $\frac{1}{2}BB$, $\frac{1}{2}AA$, $\frac{1}{2}CC$, by r_1, r_2, r_3 respectively. As in (17), the electric forces give

$$(H_1 + K_1)\log\frac{r_3}{r_2} + K_3\log\frac{r_2}{r_1} = H_2\log\frac{r_3}{r_1}. \quad \text{..............(22)}$$

The magnetic forces yield two equations, which may be regarded as expressing that the currents are the same on the two sides along BB, and that, since the section is at a negligible distance from the insulated end, there is no current in AA. Thus

$$H_1 - K_1 = -K_3 = H_2. \quad \text{.........................(23)}$$

From (22) and (23)

$$\frac{K_1}{H_1} = \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1}, \quad \text{.........................(24)}$$

$$H_2 = -K_3 = \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1}. \quad \text{.........................(25)}$$

If r_2 exceeds r_1 but little, K_1 tends to vanish, while H_2 and $-K_3$ approach unity. Again, if the radii are all great, (24), (25) reduce to

$$\frac{K_1}{H_1} = \frac{r_2 - r_1}{r_3 - r_1}, \qquad H_2 = -K_3 = \frac{r_3 - r_2}{r_3 - r_1}, \quad \text{..............(26)}$$

as already found in (8), (9).

The same method applies with but little variation to the more general problem where waves between one wire and sheath (r_1, r_1') divide so as to pass along several wires and sheaths (r_2, r_2'), (r_3, r_3'), etc., always under the condition that the whole region of irregularity is negligible in comparison with the wave-length*. The various wires and sheaths are, of course, supposed to be continuous. With a similar notation the direct and reflected waves along the first wire are denoted by H_1, K_1, and those propagated

* This condition will usually suffice. But extreme cases may be proposed where, in spite of the smallness of the intermediate region, its shape is such as to entail natural resonances of frequency agreeing with that of the principal waves. The method would then fail.

onwards along the second, third, and other wires by H_2, H_3, etc. The equations are—

$$(H_1 + K_1)\log\frac{r_1'}{r_1} = H_2\log\frac{r_2'}{r_2} = H_3\log\frac{r_3'}{r_3} = \ldots\ldots, \quad \ldots\ldots\ldots(27)$$

$$H_1 - K_1 = H_2 = H_3 = \ldots\ldots$$

It is hardly necessary to detail obvious particular cases.

The success of the method used in these problems depends upon the assumption of a great wave-length. This, of course, constitutes a limitation; but it has the advantage of eliminating the irregular motion at the junctions. In the two-dimensional examples it might be possible to pursue the approximation by determining the character of the irregular waves, at least to a certain extent, somewhat as in the question of the correction for the open end of an organ pipe.

372.

THE CORRECTION TO THE LENGTH OF TERMINATED RODS IN ELECTRICAL PROBLEMS.

[*Philosophical Magazine*, Vol. XXV. pp. 1–9, 1913.]

IN a short paper "On the Electrical Vibrations associated with thin terminated Conducting Rods"* I endeavoured to show that the difference between the half wave-length of the gravest vibration and the length (l) of the rod (of uniform section) tends to vanish relatively when the section is reduced without limit, in opposition to the theory of Macdonald which makes $\lambda = 2{\cdot}53\, l$. Understanding that the argument there put forward is not considered conclusive, I have tried to treat the question more rigorously, but the difficulties in the way are rather formidable. And this is not surprising in view of the discontinuities presented at the edges where the flat ends meet the cylindrical surface.

The problem assumes a shape simpler in some respects if we suppose that the rod of length l and radius a surrounded by a cylindrical coaxial conducting case of radius b extending to infinity in both directions. One advantage is that the vibrations are now *permanently maintained*, for no waves can escape to infinity along the tunnel, seeing that l is supposed great compared with b†. The greatness of l secures also the independence of the two ends, so that the whole correction to the length, whatever it is, may be regarded as simply the double of that due to the end of a rod infinitely long.

At an interior node of an infinitely long rod the electric forces, giving rise (we may suppose) to potential energy, are a maximum, while the magnetic forces representing kinetic energy are evanescent. The end of a terminated rod corresponds, approximately at any rate, to a node. The complications

* *Phil. Mag.* Vol. VIII. p. 105 (1904); *Scientific Papers*, Vol. V. p. 198.

† *Phil. Mag.* Vol. XLIII. p. 125 (1897); *Scientific Papers*, Vol. IV. p. 276. The conductors are supposed to be *perfect*.

due to the end thus tell mainly upon the electric forces*, and the problem is reduced to the electrostatical one of finding the *capacity* of the terminated rod as enclosed in the infinite cylindrical case at potential zero. But this simplified form of the problem still presents difficulties.

Taking cylindrical coordinates z, r, we identify the axis of symmetry with that of z, supposing also that the origin of z coincides with the flat end of the interior conducting rod which extends from $-\infty$ to 0. The enclosing case on the other hand extends from $-\infty$ to $+\infty$. At a distance from the end on the negative side the potential V, which is supposed to be unity on the rod and zero on the case, has the form

$$V_0 = \frac{\log b/r}{\log b/a}, \quad \text{...............................(1)}$$

and the *capacity* per unit length is $1/(2 \log b/a)$.

On the plane $z=0$ the value of V from $r=0$ to $r=a$ is unity. If we knew also the value of V from $r=a$ to $r=b$, we could treat separately the problems arising on the positive and negative sides. On the positive side we could express the solution by means of the functions appropriate to the complete cylinder $r<b$, and on the negative side by those appropriate to the annual cylindrical space $b>r>a$. If we assume an arbitrary value for V over the part in question of the plane $z=0$, the criterion of its suitability may be taken to be the equality of the resulting values of dV/dz on the two sides.

We may begin by supposing that (1) holds good on the negative side throughout; and we have then to form for the positive side a function which shall agree with this at $z=0$. The general expression for a function which shall vanish when $r=b$ and when $z=+\infty$, and also satisfy Laplace's equation, is

$$A_1 J_0(k_1 r)\, e^{-k_1 z} + A_2 J_0(k_2 r)\, e^{-k_2 z} + \ldots, \quad \text{..................(2)}$$

where k_1, k_2, &c. are the roots of $J_0(kb)=0$; and this is to be identified when $z=0$ with (1) from a to b and with unity from 0 to a. The coefficients A are to be found in the usual manner by multiplication with $J_0(k_n r)$ and integration over the area of the circle $r=b$. To this end we require

$$\int_0^a J_0(kr)\, r\, dr = -\frac{a}{k} J_0'(ka), \quad \text{..(3)}$$

$$\int_a^b J_0(kr)\, r\, dr = -\frac{1}{k}\{bJ_0'(kb) - aJ_0'(ka)\}, \quad \text{............................(4)}$$

$$\int_a^b \log r\, J_0(kr)\, r\, dr = -\frac{1}{k}\{b \log b\, J_0'(kb) - a \log a\, J_0'(ka)\} - \frac{1}{k^2} J_0(ka). \quad \text{... (5)}$$

* Compare the analogous acoustical questions in *Theory of Sound*, §§ 265, 317.

Thus altogether

$$\frac{J_0(ka)}{k^2 \log b/a} = A \int_a^b J_0^2(kr)\, r\, dr = \tfrac{1}{2} b^2 A J_0'^2(kb). \quad \text{..........(6)}$$

For $J_0'^2$ we may write J_1^2; so that if in (2) we take

$$A = \frac{2J_0(ka)}{k^2 b^2 J_1^2(kb) \log b/a}, \quad \text{......................(7)}$$

we shall have a function which satisfies the necessary conditions, and at $z = 0$ assumes the value 1 from 0 to a and that expressed in (1) from a to b. But the values of dV/dz are not the same on the two sides.

If we call the value, so determined on the positive as well as upon the negative side, V_0, we may denote the true value of V by $V_0 + V'$. The conditions for V' will then be the satisfaction of Laplace's equation throughout the dielectric (except at $z = 0$), that on the negative side it make $V' = 0$ both when $r = a$ and when $r = b$, and vanish at $z = -\infty$, and on the positive side $V' = 0$ when $r = b$ and when $z = +\infty$, and that when $z = 0$ V' assume the same value on the two sides between a and b and on the positive side the value zero from 0 to a. A further condition for the exact solution is that dV/dz, or $dV_0/dz + dV'/dz$, shall be the same on the two sides from $r = a$ to $r = b$ when $z = 0$.

Now whatever may be in other respects the character of V' on the negative side, it can be expressed by the series

$$V' = H_1 \phi(h_1 r) e^{h_1 z} + H_2 \phi(h_2 r) e^{h_2 z} + \ldots, \quad \text{..............(8)}$$

where $\phi(h_1 r)$, &c. are the normal functions appropriate to the symmetrical vibrations of an annular membrane of radii a and b, so that $\phi(hr)$ vanishes for $r = a$, $r = b$. In the usual notation we may write

$$\phi(hr) = \frac{J_0(hr)}{J_0(ha)} - \frac{Y_0(hr)}{Y_0(ha)}, \quad \text{......................(9)}$$

with the further condition

$$Y_0(ha) J_0(hb) - J_0(ha) Y_0(hb) = 0, \quad \text{................(10)}$$

determining the values of h. The function ϕ satisfies the same differential equation as do J_0 and Y_0.

Considering for the present only one term of the series (8), we have to find for the positive side a function which shall satisfy the other necessary conditions and when $z = 0$ make $V' = 0$ from 0 to a, and $V' = H\phi(hr)$ from a to b. As before, such a function may be expressed by

$$V' = B_1 J_0(k_1 r) e^{-k_1 z} + B_2 J_0(k_2 r) e^{-k_2 z} + \ldots, \quad \text{............(11)}$$

and the only remaining question is to find the coefficients B. For this purpose we require to evaluate

$$\int_b^a \phi(hr) J_0(kr)\, r\, dr.$$

From the differential equation satisfied by J_0 and ϕ we get

$$k^2 \int_a^b J_0(kr)\,\phi(hr)\,r\,dr = -\left[r\,.\,\phi\,.\,\frac{dJ_0}{dr}\right]_a^b + \int_a^b \frac{d\phi}{dr}\frac{dJ_0}{dr}\,r\,dr,$$

and
$$h^2 \int_a^b J_0(kr)\,\phi(hr)\,r\,dr = -\left[r\,.\,J_0\,.\,\frac{d\phi}{dr}\right]_a^b + \int_a^b \frac{d\phi}{dr}\frac{dJ_0}{dr}\,r\,dr\,;$$

so that

$$(k^2 - h^2)\int_a^b J_0(kr)\,\phi(hr)\,r\,dr = \left[rJ_0\frac{d\phi}{dr} - r\frac{dJ_0}{dr}\phi\right]_a^b$$
$$= -\,haJ_0(ka)\,\phi'(ha), \quad \text{.........(12)}$$

since here $\phi(ha) = \phi(hb) = 0$, and also $J_0(kb) = 0$. Thus in (11), corresponding to a single term of (8),

$$B = \frac{2haHJ_0(ka)\,\phi'(ha)}{(h^2 - k^2)\,b^2J_1^2(kb)}. \quad \text{.....................(13)}$$

The exact solution demands the inclusion in (8) of all the admissible values of h, with addition of (1) which in fact corresponds to a zero value of h. And each value of h contributes a part to each of the infinite series of coefficients B, needed to express the solution on the positive side.

But although an exact solution would involve the whole series of values of h, approximate methods may be founded upon the use of a limited number of them. I have used this principle in calculations relating to the potential from 1870 onwards*. A potential V, given over a closed surface, makes

$$\frac{1}{4\pi}\iiint\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\}dx\,dy\,dz, \quad \text{...........(14)}$$

reckoned over the whole included volume, a minimum. If an expression for V, involving a finite or infinite number of coefficients, is proposed which satisfies the surface condition and is such that it necessarily includes the true form of V, we may approximate to the value of (14), making it a minimum by variation of the coefficients, even though only a limited number be included. Every fresh coefficient that is included renders the approximation closer, and as near an approach as we please to the truth may be arrived at by continuing the process. The true value of (14) is equal by Green's theorem to

$$\frac{1}{4\pi}\iint V\frac{dV}{dn}\,dS, \quad \text{...........................(15)}$$

the integration being over the surface, so that at all stages of the approximation the calculated value of (14) exceeds the true value of (15). In the application to a *condenser*, whose armatures are at potentials 0 and 1,

* *Phil. Trans.* Vol. CLXI. p. 77 (1870); *Scientific Papers*, Vol. I. p. 33. *Phil. Mag.* Vol. XLIV. p. 328 (1872); *Scientific Papers*, Vol. I. p. 140. Compare also *Phil. Mag.* Vol. XLVII. p. 566 (1899), Vol. XXII. p. 225 (1911).

(15) represents the *capacity*. A calculation of capacity founded upon an approximate value of V in (14) is thus always an overestimate.

In the present case we may substitute (15) for (14), if we consider the positive and negative sides separately, since it is only at $z = 0$ that Laplace's equation fails to receive satisfaction. The complete expression for V on the right is given by combination of (2) and (11), and the surface of integration is composed of the cylindrical wall $r = b$ from $z = 0$ to $z = \infty$, and of the plane $z = 0$ from $r = 0$ to $r = b$*. The cylindrical wall contributes nothing, since V vanishes along it. At $z = 0$

$$V = \Sigma (A + B) J_0(kr), \quad -dV/dz = \Sigma k (A + B) J_0(kr);$$

and

$$(15) = \tfrac{1}{2} b^2 \Sigma k (A + B)^2 J_1^2(kb). \quad \text{.................(16)}$$

On the left the complete value of V includes (1) and (8). There are here two cylindrical surfaces, but $r = b$ contributes nothing for the same reason as before. On $r = a$ we have $V = 1$ and

$$-\frac{dV}{dr} = \frac{1}{a \log b/a} - \Sigma h H \phi'(ha) e^{hz};$$

so that this part of the surface, extending to a great distance $z = -l$, contributes to (15)

$$\frac{1}{2 \log b/a} - \frac{a}{2} \Sigma H \phi'(ha). \quad \text{.......................(17)}$$

There remains to be considered the annular area at $z = 0$. Over this

$$V = \frac{\log b/r}{\log b/a} + \Sigma H \phi(hr). \quad \text{.......................(18)}$$

$$dV/dz = \Sigma h H \phi(hr). \quad \text{..............................(19)}$$

The integrals required are

$$\int_a^b \phi(hr)\, r\, dr = -h^{-1}\{b\phi'(hb) - a\phi'(ha)\}, \quad \text{..................(20)}$$

$$\int_a^b \log r\, \phi(hr)\, r\, dr = -h^{-1}\{b \log b\, \phi'(hb) - a \log a\, \phi'(ha)\}, \quad \text{...(21)}$$

$$\int_a^b \{\phi(hr)\}^2 r\, dr = \tfrac{1}{2} b^2 \{\phi'(hb)\}^2 - \tfrac{1}{2} a^2 \{\phi'(ha)\}^2; \quad \text{..............(22)}$$

and we get for this part of the surface

$$\tfrac{1}{2} a \Sigma H \phi'(ha) + \tfrac{1}{4} \Sigma h H^2 [b^2 \{\phi'(hb)\}^2 - a^2 \{\phi'(ha)\}^2]. \quad \text{......(23)}$$

Thus for the whole surface on the left

$$(15) = \frac{1}{2 \log b/a} + \tfrac{1}{4} \Sigma h H^2 [b^2 \phi'^2(hb) - a^2 \phi'^2(ha)], \quad \text{........(24)}$$

* The surface at $z = +\infty$ may evidently be disregarded.

the simplification arising from the fact that (1) is practically a member of the series ϕ.

The calculated capacity, an overestimate unless all the coefficients H are correctly assigned, is given by addition of (16) and (24). The first approximation is obtained by omitting all the quantities H, so that the B's vanish also. The additional capacity, derived entirely from (16), is then $\frac{1}{4}b^2 \Sigma k A^2 J_1^2(kb)$, or on introduction of the value of A,

$$\frac{b}{\log^2 b/a} \Sigma \frac{J_0^2(ka)}{k^3 b^3 J_1^2(kb)}, \quad \text{.......................(25)}$$

the summation extending to all the roots of $J_0(kb) = 0$. Or if we express the result in terms of the correction δl to the length (for one end), we have

$$\delta l = \frac{2b}{\log b/a} \Sigma \frac{J_0^2(ka)}{k^2 b^3 J_1^2(kb)}, \quad \text{.....................(26)}$$

as the first approximation to δl and an overestimate.

The series in (26) converges sufficiently. $J_0^2(ka)$ is less than unity. The mth root of $J_0(x) = 0$ is $x = (m - \frac{1}{4})\pi$ approximately, and $J_1^2(x) = 2/\pi x$, so that when m is great

$$\frac{1}{x^3 J_1^2(x)} = \frac{8}{\pi (4m-1)^2}. \quad \text{........................(27)}$$

The values of the reciprocals of $x^3 J_1^2(x)$ for the earlier roots can be calculated from the tables* and for the higher roots from (27). I find

m	x	$\pm J_1(x)$	$x^{-3} \div J_1^2(x)$
1	2·4048	·51915	·2668
2	5·5201	·34027	·0513
3	8·6537	·27145	·0209
4	11·7915	·23245	·0113
5	14·9309	·20655	·0070

The next five values are ·0048, ·0035, ·0026, ·0021, ·0017. Thus for any value of a the series in (26) is

$$\cdot 2668\, J_0^2(2\cdot 405\, a/b) + \cdot 0513\, J_0^2(5\cdot 520\, a/b) + \ldots; \quad \text{......(28)}$$

it can be calculated without difficulty when a/b is given. When a/b is very small, the J's in (28) may be omitted, and we have simply to sum the numbers in the fourth column of the table and its continuation. The first ten roots give ·3720. The remainder I estimate at ·015, making in all ·387. Thus in this case

$$\delta l = \frac{\cdot 774 b}{\log b/a}. \quad \text{............................(29)}$$

* Gray and Mathews, *Bessel's Functions*, pp. 244, 247.

It is particularly to be noticed that although (29) is an overestimate, it vanishes when a tends to zero.

The next step in the approximation is the inclusion of H_1 corresponding to the first root h_1 of $\phi(hb) = 0$. For a given k, B has only one term, expressed by (13) when we write h_1, H_1 for h, H. In (16) when we expand $(A+B)^2$, we obtain three series of which the first involving A^2 is that already dealt with. It does not depend upon H_1. Constant factors being omitted, the second series depends upon

$$\Sigma \frac{J_0^2(ka)}{k(h_1^2-k^2)J_1^2(kb)}, \quad \text{..........................(30)}$$

and the third upon

$$\Sigma \frac{kJ_0^2(ka)}{(h_1^2-k^2)^2 J_1^2(kb)}, \quad \text{..........................(31)}$$

the summations including all admissible values of k. In (24) we have under Σ merely the single term corresponding to H_1, h_1. The sum of (16) and (24) is a quadratic expression in H_1, and is to be made a minimum by variation of that quantity.

The application of this process to the case of a very small leads to a rather curious result. It is known (*Theory of Sound*, § 213 a) that k_1^2 and h_1^2 are then nearly equal, so that the first terms of (30) and (31) are relatively large, and require a special evaluation. For this purpose we must revert to (10) in which, since ha is small,

$$Y_0(ha) = \log ha\, J_0(ha) + 2J_2(ha), \quad \text{..................(32)}$$

so that nearly enough

$$J_0(hb) = (h-k)\, bJ_0'(kb) = \frac{Y_0(hb)}{\log ha} = \frac{Y_0(kb)}{\log ka},$$

and

$$k - h = \frac{Y_0(kb)}{bJ_1(kb)\log ka}. \quad \text{..........................(33)}$$

Thus, when a is small enough, the first terms of (30) and (31) dominate the others, and we may take simply

$$(30) = -\frac{b\log k_1 a}{2k_1^2 Y_0(k_1 b)\, J_1(k_1 b)}, \quad \text{..................(34)}$$

$$(31) = \frac{b^2 \log^2 k_1 a}{4k_1\, Y_0^2(k_1 b)}. \quad \text{..........................(35)}$$

Also
$$\phi'(k_1 a) = -\frac{1}{k_1 a \log k_1 a}, \qquad \phi'(k_1 b) = \frac{Y_1(k_1 b)}{\log k_1 a}. \quad \text{.........(36)}$$

Using these, we find from (16) and (24)

$$\frac{b}{\log^2 b/a}\Sigma\frac{1}{k^2 b^2 J_1^2(kb)} + \frac{H_1}{k_1^2 b \log b/a \,.\, Y_0(k_1 b)\, J_1(k_1 b)} + \frac{H_1^2}{4k_1\, Y_0^2(k_1 b)}$$
$$+ \frac{l}{2\log b/a} + \frac{k_1 H_1^2}{4\log^2 k_1 a}\{b^2 Y_1^2(k_1 b) - k_1^{-2}\}, \quad \text{......(37)}$$

as the expression for the capacity which is to be made a minimum. Comparing the terms in H_1^2, we see that the two last, corresponding to the negative side, vanish in comparison with the other in virtue of the large denominator $\log^2 k_1 a$. Hence approximately

$$H_1 = -\frac{2Y_0(k_1 b)}{k_1 b \log b/a \,.\, J_1(k_1 b)}, \quad \text{.....................(38)}$$

and (37) becomes

$$\frac{l}{2\log b/a} + \frac{b}{\log^2 b/a}\Sigma\frac{1}{k^3 b^3 J_1^2(kb)} - \frac{b}{\log^2 b/a}\,\frac{1}{k_1^3 b^3 J_1^2(k_1 b)} \quad \text{...... (39)}$$

when made a minimum by variation of H_1. Thus the effect of the correction depending on the introduction of H_1 is simply to wipe out the initial term of the series which represents the first approximation to the correction.

After this it may be expected that the remaining terms of the first approximation to the correction will also disappear. On examination this conjecture will be found to be verified. Under each value of k in (16) only that part of B is important for which h has the particular value which is nearly equal to k. Thus each new H annuls the corresponding member of the series in (39), so that the continuation of the process leaves us with the first term of (39) isolated. The inference is that the correction to the capacity vanishes in comparison with $b \div \log^2 b/a$, or that δl vanishes in comparison with $b \div \log b/a$. It would seem that δl is of the order $b \div \log^2 b/a$, but it would not be easy to find the numerical coefficient by the present method.

In any case the correction δl to the length of the rod vanishes in the electrostatical problem when the radius of the rod is diminished without limit—a conclusion which I extend to the vibrational problem specified in the earlier portion of this paper.

373.

ON CONFORMAL REPRESENTATION FROM A MECHANICAL POINT OF VIEW.

[*Philosophical Magazine*, Vol. xxv. pp. 698–702, 1913.]

In what is called conformal representation the coordinates of one point x, y in a plane are connected with those of the corresponding point ξ, η by the relation

$$x + iy = f(\xi + i\eta), \qquad \qquad (1)$$

where f denotes an arbitrary function. In this transformation angles remain unaltered, and corresponding infinitesimal figures are similar, though not in general similarly situated. If we attribute to ξ, η values in arithmetical progression with the same small common difference, the simple square net-work is represented by two sets of curves crossing one another at right angles so as to form what are ultimately squares when the original common difference is made small enough. For example, as a special case of (1), if

$$x + iy = c \sin(\xi + i\eta), \qquad \qquad (2)$$

$$x = c \sin \xi \cosh \eta, \quad y = c \cos \xi \sinh \eta;$$

and the curves corresponding to η = constant are

$$\frac{x^2}{c^2 \cosh^2 \eta} + \frac{y^2}{c^2 \sinh^2 \eta} = 1, \qquad \qquad (3)$$

and those corresponding to ξ = constant are

$$\frac{x^2}{c^2 \sin^2 \xi} - \frac{y^2}{c^2 \cos^2 \xi} = 1, \qquad \qquad (4)$$

a set of confocal ellipses and hyperbolas.

It is usual to refer x, y and ξ, η to separate planes and, as far as I have seen, no *transition* from the one position to the other is contemplated. But of course there is nothing to forbid the two sets of coordinates being taken in the same plane and measured on the same axes. We may then

regard the angular points of the network as moving from the one position to the other.

Some fifteen or twenty years ago I had a model made for me illustrative of these relations. The curves have their material embodiment in wires of hard steel. At the angular points the wires traverse small and rather thick brass disks, bored suitably so as to impose the required perpendicularity, the

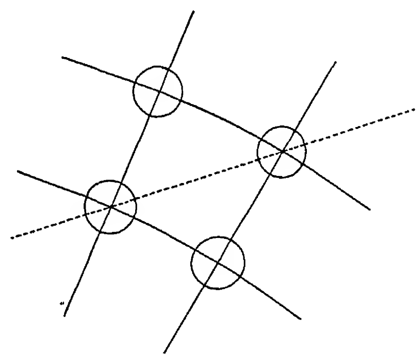


Fig. 1.

two sets of wires being as nearly as may be in the same plane. But something more is required in order to secure that the rectangular element of the network shall be *square*. To this end a third set of wires (shown dotted in fig. 1) was introduced, traversing the corner pieces through borings making 45° with the previous ones. The model answered its purpose to a certain extent, but the manipulation was not convenient on account of the friction entailed as the wires slip through the closely-fitting corner pieces. Possibly with the aid of rollers an improved construction might be arrived at.

The material existence of the corner pieces in the model suggests the consideration of a continuous two-dimensional medium, say a lamina, whose deformation shall represent the transformation. The lamina must be of such a character as absolutely to preclude *shearing*. On the other hand, it must admit of expansion and contraction equal in all (two-dimensional) directions, and if the deformation is to persist without the aid of applied forces, such expansion must be unresisted.

Since the deformation is now regarded as taking place continuously, f in (1) must be supposed to be a function of the time t as well as of $\xi + i\eta$. We may write

$$x + iy = f(t, \xi + i\eta). \quad\quad (5)$$

The component velocities u, v of the particle which at time t occupies the position x, y are given by dx/dt, dy/dt, so that

$$u + iv = \frac{d}{dt} f(t, \xi + i\eta). \quad\quad (6)$$

Between (5) and (6) $\xi + i\eta$ may be eliminated; $u + iv$ then becomes a function of t and of $x + iy$, say

$$u + iv = F(t, x + iy). \quad \text{.........................}(7)$$

The equation with which we started is of what is called in Hydrodynamics the Lagrangian type. We follow the motion of an individual particle. On the other hand, (7) is of the Eulerian type, expressing the velocities to be found at any time at a specified place. Keeping t fixed, *i.e.* taking, as it were, an instantaneous view of the system, we see that u, v, as given by (7), satisfy

$$(d^2/dx^2 + d^2/dy^2)(u, v) = 0, \quad \text{.......................}(8)$$

equations which hold also for the irrotational motion of an incompressible liquid.

It is of interest to compare the present motion with that of a highly viscous two-dimensional fluid, for which the equations are*

$$\rho\frac{Du}{Dt} = \rho X - \frac{dp}{dx} + \mu'\frac{d\theta}{dx} + \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right),$$

$$\rho\frac{Dv}{Dt} = \rho Y - \frac{dp}{dy} + \mu'\frac{d\theta}{dy} + \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\right),$$

where $$\theta = \frac{du}{dx} + \frac{dv}{dy}.$$

If the pressure is independent of density and if the inertia terms are neglected, these equations are satisfied provided that

$$\rho X + \mu' d\theta/dx = 0, \quad \rho Y + \mu' d\theta/dy = 0.$$

In the case of real viscous fluids, there is reason to think that $\mu' = \frac{1}{3}\mu$. Impressed forces are then required so long as the fluid is moving. The supposition that p is constant being already a large departure from the case of nature, we may perhaps as well suppose $\mu' = 0$, and then no impressed bodily forces are called for either at rest or in motion.

If we suppose that the motion in (7) is *steady* in the hydrodynamical sense, $u + iv$ must be independent of t, so that the elimination of $\xi + i\eta$ between (5) and (6) must carry with it the elimination of t. This requires that df/dt in (6) be a function of f, and not otherwise of t and $\xi + i\eta$; and it follows that (5) must be of the form

$$x + iy = F_1\{t + F_2(\xi + i\eta)\}. \quad \text{.........................}(9)$$

* Stokes, *Camb. Trans.* 1850; *Mathematical and Physical Papers*, Vol. IV. p. 11. It does not seem to be generally known that the laws of dynamical similarity for viscous fluids were formulated in this memoir. Reynolds's important application was 30 years later.

where F_1, F_2 denote arbitrary functions. Another form of (9) is

$$F_3(x+iy) = t + F_2(\xi + i\eta). \quad \text{..................(10)}$$

For an individual particle $F_2(\xi + i\eta)$ is constant, say $a + ib$. The equation of the stream-line followed by this particle is obtained by equating to ib the imaginary part of $F_3(x+iy)$.

As an example of (9), suppose that

$$x + iy = c \sin\{it + \xi + i\eta\}, \quad \text{......................(11)}$$

so that $$x = c \sin \xi \,.\, \cosh(\eta + t), \quad y = c \cos \xi \,.\, \sinh(\eta + t), \quad \text{........(12)}$$

whence on elimination of t we obtain (4) as the equation of the stream-lines.

It is scarcely necessary to remark that the law of flow along the stream-lines is entirely different from that with which we are familiar in the flow of incompressible liquids. In the latter case the motion is rapid at any place where neighbouring stream-lines approach one another closely. Here, on the contrary, the motion is exceptionally slow at such a place.

374.

ON THE APPROXIMATE SOLUTION OF CERTAIN PROBLEMS RELATING TO THE POTENTIAL. II.

[*Philosophical Magazine*, Vol. XXVI. pp. 195–199, 1913.]

THE present paper may be regarded as supplementary to one with the same title published a long while ago*. In two dimensions, if ϕ, ψ be potential and stream functions, and if (e.g.) ψ be zero along the line $y=0$, we may take

$$\phi = \int f\,dx - \frac{y^2}{1\,.\,2}f'' + \frac{y^4}{1\,.\,2\,.\,3\,.\,4}f'''' - \ldots \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\psi = yf - \frac{y^3}{1\,.\,2\,.\,3}f'' + \frac{y^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}f^{\mathrm{iv}} - \ldots \quad \ldots\ldots\ldots\ldots(2)$$

f being a function of x so far arbitrary. These values satisfy the general conditions for the potential and stream functions, and when $y=0$ make

$$d\phi/dx = f, \qquad \psi = 0.$$

Equation (2) may be regarded as determining the lines of flow (any one of which may be supposed to be the boundary) in terms of f. Conversely, if y be supposed known as a function of x and ψ be constant (say unity), we may find f by successive approximation. Thus

$$f = \frac{1}{y} + \frac{y^2}{6}\frac{d^2}{dx^2}\left(\frac{1}{y}\right) + \frac{y^2}{36}\frac{d^2}{dx^2}\left\{y^2\frac{d^2}{dx^2}\left(\frac{1}{y}\right)\right\} - \frac{y^4}{120}\frac{d^4}{dx^4}\left(\frac{1}{y}\right) \ldots\ldots(3)$$

We may use these equations to investigate the stream-lines for which ψ has a value intermediate between 0 and 1. If η denote the corresponding value of y, we have to eliminate f between

$$1 = yf - \frac{y^3}{6}f'' + \frac{y^5}{120}f^{\mathrm{iv}} - \ldots,$$

and

$$\psi = \eta f - \frac{\eta^3}{6}f'' + \frac{\eta^5}{120}f^{\mathrm{iv}} - \ldots;$$

whence

$$\eta = \psi y + \frac{f''}{6}(y\eta^3 - \eta y^3) - \frac{f^{\mathrm{iv}}}{120}(y\eta^5 - \eta y^5),$$

* *Proc. Lond. Math. Soc.* Vol. VII. p. 75 (1876); *Scientific Papers*, Vol. I. p. 272.

or by use of (3)

$$\eta = \psi y + \frac{y^4(\psi^3-\psi)}{6}\frac{d^2}{dx^2}\left(\frac{1}{y}\right) + \frac{y^7(\psi^2-1)(3\psi^2-\psi)}{36}\left\{\frac{d^2}{dx^2}\left(\frac{1}{y}\right)\right\}^2$$

$$+ \frac{y^4(\psi^3-\psi)}{36}\frac{d^2}{dx^2}\left\{y^2\frac{d^2}{dx^2}\left(\frac{1}{y}\right)\right\} - \frac{y^6(\psi^5-\psi)}{120}\frac{d^4}{dx^4}\left(\frac{1}{y}\right). \quad \ldots\ldots(4)$$

The evanescence of ψ when $y=0$ may arise from this axis being itself a boundary, or from the second boundary being a symmetrical curve situated upon the other side of the axis. In the former paper expressions for the "resistance" and "conductivity" were developed.

We will now suppose that $\psi=0$ along a *circle* of radius a, in substitution for the axis of x. Taking polar coordinates $a+r$ and θ, we have as the general equation

$$(a+r)^2\frac{d^2\psi}{dr^2} + (a+r)\frac{d\psi}{dr} + \frac{d^2\psi}{d\theta^2} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Assuming
$$\psi = R_1 r + R_2 r^2 + R_3 r^3 + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where R_1, R_2, &c., are functions of θ, we find on substitution in (5)

$$\left.\begin{aligned} 2a^2R_2 + aR_1 &= 0,\\ 6a^2R_3 + 6aR_2 + R_1 + R_1'' &= 0;\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

so that
$$\psi = R_1 r - \frac{R_1 r^2}{2a} + \frac{(2R_1 - R_1'')r^3}{6a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

is the form corresponding to (2) above.

If $\psi=1$, (8) yields

$$R_1 = \frac{1}{r} + \frac{1}{2a} - \frac{r^2}{12a^2} + \frac{r^2}{6a^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

expressing R_1 as a function of θ, when r is known as such. To interpolate a curve for which ρ takes the place of r, we have to eliminate R_1 between

$$1 = R_1 r - \frac{R_1 r^2}{2a} + \frac{(2R_1 - R_1'')r^2}{6a^2},$$

and
$$\psi = R_1\rho - \frac{R_1\rho^2}{2a} + \frac{(2R_1 - R_1'')\rho^2}{6a^2}.$$

Thus
$$\rho = r\psi - \frac{R_1}{2a}(\rho r^2 - r\rho^2) + \frac{2R_1 - R_1''}{6a^2}(\rho r^3 - r\rho^3),$$

and by successive approximation with use of (9)

$$\rho = r\psi + \frac{r^2}{a}\frac{\psi(\psi-1)}{1.2} + \frac{r^3}{a^2}\frac{\psi(\psi-1)(\psi-2)}{1.2.3} + \frac{r^4}{a^2}\frac{\psi(\psi^2-1)}{6}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right). \quad (10)$$

The significance of the first three terms is brought out if we suppose that r is constant (α), so that the last term vanishes. In this case the exact solution is

$$\log \frac{a+\rho}{a} = \psi \log \frac{a+\alpha}{a}, \quad \text{......................(11)}$$

whence

$$\frac{\rho}{a} = \left(\frac{a+\alpha}{a}\right)^{\psi} - 1 = \psi \frac{\alpha}{a} + \frac{\psi(\psi-1)}{1.2}\frac{\alpha^2}{a^2} + \frac{\psi(\psi-1)(\psi-2)}{1.2.3}\frac{\alpha^3}{a^3} + \dots \quad \text{...(12)}$$

in agreement with (10).

In the above investigation ψ is supposed to be zero exactly upon the circle of radius a. If the circle whose centre is taken as origin of coordinates be merely the circle of curvature of the curve $\psi = 0$ at the point ($\theta = 0$) under consideration, ψ will not vanish exactly upon it, but only when r has the approximate value $c\theta^3$, c being a constant. In (6) an initial term R_0 must be introduced, whose approximate value is $-c\theta^3 R_1$. But since R_0'' vanishes with θ, equation (7) and its consequences remain undisturbed and (10) is still available as a formula of interpolation. In all these cases, the success of the approximation depends of course upon the degree of slowness with which y, or r, varies.

Another form of the problem arises when what is given is not a pair of neighbouring curves along each of which (*e.g.*) the stream-function is constant, but *one* such curve together with the variation of potential along it. It is then required to construct a neighbouring stream-line and to determine the distribution of potential upon it, from which again a fresh departure may be made if desired. For this purpose we regard the rectangular coordinates x, y as functions of ξ (potential) and η (stream-function), so that

$$x + iy = f(\xi + i\eta), \quad \text{..........................(13)}$$

in which we are supposed to know $f(\xi)$ corresponding to $\eta = 0$, *i.e.*, x and y are there known functions of ξ. Take a point on $\eta = 0$, at which without loss of generality ξ may be supposed also to vanish, and form the expressions for x and y in the neighbourhood. From

$$x + iy = A_0 + iB_0 + (A_1 + iB_1)(\xi + i\eta) + (A_2 + iB_2)(\xi + i\eta)^2 + \dots,$$

we derive

$$\begin{aligned} x = A_0 &+ A_1\xi - B_1\eta + A_2(\xi^2 - \eta^2) - 2B_2\xi\eta \\ &+ A_3(\xi^3 - 3\xi\eta^2) - B_3(3\xi^2\eta - \eta^3) \\ &+ A_4(\xi^4 - 6\xi^2\eta^2 + \eta^4) - 4B_4(\xi^3\eta - \xi\eta^3) + \dots, \end{aligned}$$

$$\begin{aligned} y = B_0 &+ B_1\xi + A_1\eta + 2A_2\xi\eta + B_2(\xi^2 - \eta^2) \\ &+ A_3(3\xi^2\eta - \eta^3) + B_3(\xi^3 - 3\xi\eta^2) \\ &+ 4A_4(\xi^3\eta - \xi\eta^3) + B_4(\xi^4 - 6\xi^2\eta^2 + \eta^4) + \dots. \end{aligned}$$

When $\eta = 0$,

$$x = A_0 + A_1\xi + A_2\xi^2 + A_3\xi^3 + A_4\xi^4 + \dots,$$

$$y = B_0 + B_1\xi + B_2\xi^2 + B_3\xi^3 + B_4\xi^4 + \dots.$$

Since x and y are known as functions of ξ when $\eta=0$, these equations determine the A's and the B's, and the general values of x and y follow. When $\xi=0$, but η undergoes an increment,

$$x = A_0 - B_1\eta - A_2\eta^2 + B_3\eta^3 + A_4\eta^4 - \ldots, \qquad \text{(14)}$$

$$y = B_0 + A_1\eta - B_2\eta^2 - A_3\eta^3 + B_4\eta^4 + \ldots, \qquad \text{(15)}$$

in which we may suppose $\eta=1$.

The A's and B's are readily determined if we know the values of x and y for $\eta=0$ and for equidistant values of ξ, say $\xi=0$, $\xi=\pm 1$, $\xi=\pm 2$. Thus, if the values of x be called x_0, x_{-1}, x_1, x_2, x_{-2}, we find

$$A_0 = x_0, \qquad \text{and}$$

$$A_1 = \frac{2}{3}(x_1 - x_{-1}) - \frac{1}{12}(x_2 - x_{-2}), \quad A_3 = \frac{x_2 - x_{-2}}{12} - \frac{x_1 - x_{-1}}{6},$$

$$A_2 = \frac{2(x_1 + x_{-1} - 2x_0)}{3} - \frac{x_2 + x_{-2} - 2x_0}{24},$$

$$A_4 = \frac{x_2 + x_{-2} - 2x_0}{24} - \frac{x_1 + x_{-1} - 2x_0}{6}.$$

The B's are deduced from the A's by merely writing y for x throughout. Thus from (14) when $\xi=0$, $\eta=1$,

$$x = x_0 - \frac{5}{6}(x_1 + x_{-1} - 2x_0) + \frac{1}{12}(x_2 + x_{-2} - 2x_0)$$
$$- \frac{5}{6}(y_1 - y_{-1}) + \frac{1}{6}(y_2 - y_{-2}). \qquad \text{(16)}$$

Similarly

$$y = y_0 - \frac{5}{6}(y_1 + y_{-1} - 2y_0) + \frac{1}{12}(y_2 + y_{-2} - 2y_0)$$
$$+ \frac{5}{6}(x_1 - x_{-1}) - \frac{1}{6}(x_2 - x_{-2}). \qquad \text{(17)}$$

By these formulæ a point is found upon a new stream-line ($\eta=1$) corresponding to a given value of ξ. And there would be no difficulty in carrying the approximation further if desired.

As an example of the kind of problem to which these results might be applied, suppose that by observation or otherwise we know the form of the upper stream-line constituting part of the free surface when liquid falls steadily over a two-dimensional weir. Since the velocity is known at every point of the free surface, we are in a position to determine ξ along this stream-line, and thus to apply the formulæ so as to find interior stream-lines in succession.

Again (with interchange of ξ and η) we could find what forms are admissible for the second coating of a two-dimensional condenser, in order that the charge upon the first coating, given in size and shape, may have a given value at every point.

[*Sept.* 1916. As another example permanent wave-forms may be noticed.]

ON THE PASSAGE OF WAVES THROUGH FINE SLITS IN THIN OPAQUE SCREENS.

[*Proceedings of the Royal Society*, A, Vol. LXXXIX. pp. 194–219, 1913.]

IN a former paper* I gave solutions applicable to the passage of light through very narrow slits in infinitely thin perfectly opaque screens, for the two principal cases where the polarisation is either parallel or perpendicular to the length of the slit. It appeared that if the width ($2b$) of the slit is very small in comparison with the wave-length (λ), there is a much more free passage when the electric vector is perpendicular to the slit than when it is parallel to the slit, so that unpolarised light incident upon the screen will, after passage, appear polarised in the former manner. This conclusion is in accordance with the observations of Fizeau† upon the very narrowest slits. Fizeau found, however, that somewhat wider slits (scratches upon silvered glass) gave the opposite polarisation; and I have long wished to extend the calculations to slits of width comparable with λ. The subject has also a practical interest in connection with observations upon the Zeeman effect‡.

The analysis appropriate to problems of this sort would appear to be by use of elliptic coordinates; but I have not seen my way to a solution on these lines, which would, in any case, be rather complicated. In default of such a solution, I have fallen back upon the approximate methods of my former paper. Apart from the intended application, some of the problems which present themselves have an interest of their own. It will be convenient to repeat the general argument almost in the words formerly employed.

* "On the Passage of Waves through Apertures in Plane Screens and Allied Problems," *Phil. Mag.* 1897, Vol. XLIII. p. 259; *Scientific Papers*, Vol. IV. p. 283.

† *Annales de Chimie*, 1861, Vol. LXIII. p. 385; Mascart's *Traité d'Optique*, § 645. See also *Phil. Mag.* 1907, Vol. XIV. p. 350; *Scientific Papers*, Vol. V. p. 417.

‡ Zeeman, *Amsterdam Proceedings*, October, 1912.

Plane waves of simple type impinge upon a parallel screen. The screen is supposed to be infinitely thin and to be perforated by some kind of aperture. Ultimately, one or both dimensions of the aperture will be regarded as small, or, at any rate, as not large, in comparison with the wave-length (λ); and the investigation commences by adapting to the present purpose known solutions concerning the flow of incompressible fluids.

The functions that we require may be regarded as velocity-potentials ϕ, satisfying

$$d^2\phi/dt^2 = V\nabla^2\phi, \qquad \text{.............................(1)}$$

where $$\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2,$$

and V is the velocity of propagation. If we assume that the vibration is everywhere proportional to e^{int}, (1) becomes

$$(\nabla^2 + k^2)\,\phi = 0, \qquad \text{.............................(2)}$$

where $$k = n/V = 2\pi/\lambda. \qquad \text{.............................(3)}$$

It will conduce to brevity if we suppress the factor e^{int}. On this understanding the equation of waves travelling parallel to x in the positive direction, and accordingly incident upon the negative side of the screen situated at $x = 0$, is

$$\phi = e^{-ikx}. \qquad \text{.................................(4)}$$

When the solution is complete, the factor e^{int} is to be restored, and the imaginary part of the solution is to be rejected. The realised expression for the incident waves will therefore be

$$\phi = \cos(nt - kx). \qquad \text{.............................(5)}$$

There are two cases to be considered corresponding to two alternative boundary conditions. In the first (i) $d\phi/dn = 0$ over the unperforated part of the screen, and in the second (ii) $\phi = 0$. In case (i) dn is drawn outwards normally, and if we take the axis of z parallel to the length of the slit, ϕ will represent the magnetic component parallel to z, usually denoted by c, so that this case refers to vibrations for which the electric vector is perpendicular to the slit. In the second case (ii) ϕ is to be identified with the component parallel to z of the electric vector R, which vanishes upon the walls, regarded as perfectly conducting. We proceed with the further consideration of case (i).

If the screen be complete, the reflected waves under condition (i) have the expression $\phi = e^{ikx}$. Let us divide the actual solution into two parts, χ and ψ; the first, the solution which would obtain were the screen complete; the second, the alteration required to take account of the aperture; and let us distinguish by the suffixes m and p the values applicable upon the negative (*minus*), and upon the positive side of the screen. In the present case we have

$$\chi_m = e^{-ikx} + e^{ikx}, \qquad \chi_p = 0. \qquad \text{.......................(6)}$$

This χ-solution makes $d\chi_m/dn=0$, $d\chi_p/dn=0$ over the whole plane $x=0$, and over the same plane $\chi_m=2$, $\chi_p=0$.

For the supplementary solution, distinguished in like manner upon the two sides, we have

$$\psi_m=\iint \Psi_m \frac{e^{-ikr}}{r}\, dS, \qquad \psi_p=\iint \Psi_p \frac{e^{-ikr}}{r}\, dS, \quad \text{.............(7)}$$

where r denotes the distance of the point at which ψ is to be estimated from the element dS of the aperture, and the integration is extended over the whole of the area of aperture. Whatever functions of position Ψ_m, Ψ_p may be, these values on the two sides satisfy (2), and (as is evident from symmetry) they make $d\psi_m/dn$, $d\psi_p/dn$ vanish over the wall, viz., the unperforated part of the screen, so that the required condition over the wall for the complete solution is already satisfied. It remains to consider the further conditions that ϕ and $d\phi/dx$ shall be continuous across the *aperture*. These conditions require that on the aperture

$$2+\psi_m=\psi_p, \qquad d\psi_m/dx=d\psi_p/dx. \quad \text{..............(8)*}$$

The second is satisfied if $\Psi_p=-\Psi_m$; so that

$$\psi_m=\iint \Psi_m \frac{e^{-ikr}}{r}\, dS, \qquad \psi_p=-\iint \Psi_m \frac{e^{-ikr}}{r}\, dS, \quad \text{...........(9)}$$

making the values of ψ_m, ψ_p equal and opposite at all corresponding points, viz., points which are images of one another in the plane $x=0$. In order further to satisfy the first condition, it suffices that over the area of aperture

$$\psi_m=-1, \qquad \psi_p=1, \quad \text{......................(10)}$$

and the remainder of the problem consists in so determining Ψ_m that this shall be the case.

It should be remarked that Ψ in (9) is closely connected with the normal velocity at dS. In general,

$$\frac{d\psi}{dx}=\iint \Psi \frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right) dS. \quad \text{......................(11)}$$

At a point (x) infinitely close to the surface, only the neighbouring elements contribute to the integral, and the factor e^{-ikr} may be omitted. Thus

$$\frac{d\psi}{dx}=-\iint \Psi \frac{x}{r^2}\, dS=-2\pi x\int_x^\infty \Psi\frac{r\,dr}{r^3}=-2\pi\Psi; \quad \text{or} \quad \Psi=-\frac{1}{2\pi}\frac{d\psi}{dn}, \quad \text{...(12)}$$

$d\psi/dn$ being the normal velocity at the point of the surface in question.

* The use of dx implies that the variation is in a fixed direction, while dn may be supposed to be drawn outwards from the screen in both cases.

In the original paper these results were applied to an aperture, especially of elliptical form, whose dimensions are small in comparison with λ. For our present purpose we may pass this over and proceed at once to consider the case where the aperture is an infinitely long slit with parallel edges, whose width is small, or at the most comparable with λ.

The velocity-potential of a point-source, viz., e^{-ikr}/r, is now to be replaced by that of a linear source, and this, in general, is much more complicated. If we denote it by $D(kr)$, r being the distance from the line of the point where the potential is required, the expressions are*

$$D(kr) = -\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}\left\{1 - \frac{1^2}{1\,.\,8ikr} + \frac{1^2\,.\,3^2}{1\,.\,2\,.\,(8ikr)^2} - \dots\right\}$$

$$= \left(\gamma + \log\frac{ikr}{2}\right)\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2\,.\,4^2} - \dots\right\}$$

$$+ \frac{k^2r^2}{2^2}S_1 - \frac{k^4r^4}{2^2\,.\,4^2}S_2 + \frac{k^6r^6}{2^2\,.\,4^2\,.\,6^2}S_3 - \dots, \qquad (13)$$

where γ is Euler's constant (0·577215), and

$$S_m = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + 1/m. \qquad (14)$$

Of these the first is "semi-convergent" and is applicable when kr is large; the second is fully convergent and gives the form of the function when kr is moderate. The function D may be regarded as being derived from e^{-ikr}/r by integration over an infinitely long and infinitely narrow strip of the surface S.

As the present problem is only a particular case, equations (6) and (10) remain valid, while (9) may be written in the form

$$\psi_m = \int \Psi_m D(kr)\,dy, \qquad \psi_p = -\int \Psi_m D(kr)\,dy, \qquad (15)$$

the integrations extending over the width of the slit from $y = -b$ to $y = +b$. It remains to determine Ψ_m, so that on the aperture $\psi_m = -1$, $\psi_p = +1$.

At a sufficient distance from the slit, supposed for the moment to be very narrow, $D(kr)$ may be removed from under the integral sign and also be replaced by its limiting form given in (13). Thus

$$\psi_m = -\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}\int \Psi_m\,dy. \qquad (16)$$

If the slit be not very narrow, the partial waves arising at different parts of the width will arrive in various phases, of which due account must be taken. The disturbance is no longer circularly symmetrical as in (16). But if, as is usual in observations with the microscope, we restrict ourselves to

* See *Theory of Sound*, § 341.

the direction of original propagation, equality of phase obtains, and (16) remains applicable even in the case of a wide slit. It only remains to determine Ψ_m as a function of y, so that for all points upon the aperture

$$\int_{-b}^{+b} \Psi_m D(kr)\, dy = -1, \qquad \text{.......................(17)}$$

where, since kr is supposed moderate throughout, the second form in (13) may be employed.

Before proceeding further it may be well to exhibit the solution, as formerly given, for the case of a very narrow slit. Interpreting ϕ as the velocity-potential of aerial vibrations and having regard to the known solution for the flow of incompressible fluid through a slit in an infinite plane wall, we may infer that Ψ_m will be of the form $A(b^2 - y^2)^{-\frac{1}{2}}$, where A is some constant. Thus (17) becomes

$$A\left[(\gamma + \log \tfrac{1}{2} ik)\,\pi + \int_{-b}^{+b} \frac{\log r \,.\, dy}{\sqrt{(b^2 - y^2)}}\right] = -1. \qquad \text{...........(18)}$$

In this equation the first part is obviously independent of the position of the point chosen, and if the form of Ψ_m has been rightly taken the second integral must also be independent of it. If its coordinate be η, lying between $\pm b$,

$$\int_{-b}^{+b} \frac{\log r \,.\, dy}{\sqrt{(b^2 - y^2)}} = \int_{-b}^{\eta} \frac{\log(\eta - y)\, dy}{\sqrt{(b^2 - y^2)}} + \int_{\eta}^{b} \frac{\log(y - \eta)\, dy}{\sqrt{(b^2 - y^2)}} \qquad \text{......(19)}$$

must be independent of η. To this we shall presently return; but merely to determine A in (18) it suffices to consider the particular case of $\eta = 0$. Here

$$\int_{-b}^{+b} \frac{\log r \,.\, dy}{\sqrt{(b^2 - y^2)}} = 2\int_0^b \frac{\log y \,.\, dy}{\sqrt{(b^2 - y^2)}} = 2\int_0^{\frac{1}{2}\pi} \log(b \cos\theta)\, d\theta = \pi \log(\tfrac{1}{2} b).$$

Thus $\qquad A(\gamma + \log \tfrac{1}{4} ikb)\,\pi = -1, \quad$ and $\quad \int_{-b}^{+b} \Psi_m dy = \pi A;$

so that (16) becomes $\qquad \psi_m = \dfrac{e^{-ikr}}{\gamma + \log(\frac{1}{4} ikb)} \left(\dfrac{\pi}{2ikr}\right)^{\frac{1}{2}}. \qquad$ ....(20)

From this, ψ_p is derived by simply prefixing a negative sign.

The realised solution is obtained from (20) by omitting the imaginary part after introduction of the suppressed factor e^{int}. If the imaginary part of $\log(\frac{1}{4} ikb)$ be neglected, the result is

$$\psi_m = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \frac{\cos(nt - kr - \frac{1}{4}\pi)}{\gamma + \log(\frac{1}{4} kb)}, \qquad \text{.................(21)}$$

corresponding to $\qquad \chi_m = 2 \cos nt \, \cos kx. \qquad$ ....(22)

Perhaps the most remarkable feature of the solution is the very limited dependence of the transmitted vibration on the *width* $(2b)$ of the aperture.

We will now verify that (19) is independent of the special value of η. Writing $y = b\cos\theta$, $\eta = b\cos\alpha$, we have

$$\int_{-b}^{+b} \frac{\log r \, . \, dy}{\sqrt{(b^2 - y^2)}} = \int_0^{\pi} \log(\tfrac{1}{2}b)\, d\theta + \int_0^{\alpha} \log 2(\cos\theta - \cos\alpha)\, d\theta$$
$$+ \int_{\alpha}^{\pi} \log 2(\cos\alpha - \cos\theta)\, d\theta = \pi \log(\tfrac{1}{2}b)$$
$$+ \int_0^{\pi} \log\left\{2\sin\frac{\alpha+\theta}{2}\right\} d\theta + \int_0^{\alpha} \log\left\{2\sin\frac{\alpha-\theta}{2}\right\} d\theta + \int_{\alpha}^{\pi} \log\left\{2\sin\frac{\theta-\alpha}{2}\right\} d\theta$$
$$= \pi\log\tfrac{1}{2}b + 2\int_{\frac{1}{2}\alpha}^{\frac{1}{2}\pi+\frac{1}{2}\alpha} \log(2\sin\phi)\, d\phi + 2\int_0^{\frac{1}{2}\alpha} \log(2\sin\phi)\, d\phi$$
$$+ 2\int_0^{\frac{1}{2}\pi-\frac{1}{2}\alpha} \log(2\sin\phi)\, d\phi$$
$$= \pi\log\tfrac{1}{2}b + 2\int_0^{\frac{1}{2}\pi} \log(2\sin\phi)\, d\phi + 2\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\frac{1}{2}\alpha} \log(2\sin\phi)\, d\phi$$
$$+ 2\int_0^{\frac{1}{2}\pi-\frac{1}{2}\alpha} \log(2\sin\phi)\, d\phi$$
$$= \pi\log\tfrac{1}{2}b + 4\int_0^{\frac{1}{2}\pi} \log(2\sin\phi)\, d\phi,$$

as we see by changing ϕ into $\pi - \phi$ in the second integral. Since α has disappeared, the original integral is independent of η. In fact*

$$\int_0^{\frac{1}{2}\pi} \log(2\sin\phi)\, d\phi = 0,$$

and we have
$$\int_{-b}^{+b} \frac{\log r \, . \, dy}{\sqrt{(b^2 - y^2)}} = \pi\log\tfrac{1}{2}b, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

as in the particular case of $\eta = 0$.

The required condition (17) can thus be satisfied by the proposed form of Ψ, provided that kb be small enough. When kb is greater, the resulting value of ψ in (15) will no longer be constant over the aperture, but we may find what the actual value is as a function of η by carrying out the integration with inclusion of more terms in the series representing D. As a preliminary, it will be convenient to discuss certain definite integrals which present themselves. The first of the series, which has already occurred, we will call h_0, so that

$$h_0 = \int_0^{\frac{1}{2}\pi} \log(2\sin\theta)\, d\theta = \int_0^{\frac{1}{2}\pi} \log(2\cos\theta)\, d\theta = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \log(2\sin 2\theta)\, d\theta$$
$$= \tfrac{1}{4}\int_0^{\pi} \log(2\sin\phi)\, d\phi = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \log(2\sin\phi)\, d\phi = \tfrac{1}{2}h_0.$$

* See below.

Accordingly, $h_0 = 0$. More generally we set, n being an even integer,

$$h_n = \int_0^{\frac{1}{2}\pi} \sin^n \theta \log (2 \sin \theta)\, d\theta, \quad \text{.....................(24)}$$

or, on integration by parts,

$$h_n = \int_0^{\frac{1}{2}\pi} \cos\theta \{(n-1) \sin^{n-2}\theta \cos\theta \log (2\sin\theta) + \sin^{n-2}\theta\cos\theta\}\, d\theta$$

$$= (n-1)(h_{n-2} - h_n) + \int_0^{\frac{1}{2}\pi} (\sin^{n-2}\theta - \sin^n\theta)\, d\theta.$$

Thus $$h_n = \frac{n-1}{n} h_{n-2} + \frac{1}{n^2} \frac{n-3,\ n-5 \ldots 1}{n-2,\ n-4 \ldots 2} \frac{\pi}{2}, \quad \text{...........(25)}$$

by which the integrals h_n can be calculated in turn. Thus

$$h_2 = \pi/8,$$

$$h_4 = \frac{3}{4} h_2 + \frac{1}{4^2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{2} \frac{3 \cdot 1}{4 \cdot 2} \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} \right),$$

$$h_6 = \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \frac{\pi}{2} \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} \right) + \frac{1}{6^2} \frac{3 \cdot 1}{4 \cdot 2} \frac{\pi}{2}$$

$$= \frac{\pi}{2} \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} \right).$$

Similarly $$h_8 = \frac{\pi}{2} \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} \right),$$ and so on.

It may be remarked that the series within brackets, being equal to

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots,$$

approaches ultimately the limit log 2. A tabulation of the earlier members of the series of integrals will be convenient:—

TABLE I.

$2h_0/\pi$	$= 0$	
$2h_2/\pi$	$= 1/4$	$= 0{\cdot}25$
$2h_4/\pi$	$= 7/32$	$= 0{\cdot}21875$
$2h_6/\pi$	$= 37/192$	$= 0{\cdot}19271$
$2h_8/\pi$	$= 533/3072$	$= 0{\cdot}17350$
$2h_{10}/\pi$	$= 1627/10240$	$= 0{\cdot}15889$
$2h_{12}/\pi$	$= 18107/122880$	$= 0{\cdot}14736$
$2h_{14}/\pi$	$=$	$= 0{\cdot}13798$
$2h_{16}/\pi$	$=$	$= 0{\cdot}13018$
$2h_{18}/\pi$	$=$	$= 0{\cdot}12356$
$2h_{20}/\pi$	$=$	$= 0{\cdot}11784$

The last four have been calculated in sequence by means of (25).

In (24) we may, of course, replace $\sin\theta$ by $\cos\theta$ throughout. If both $\sin\theta$ and $\cos\theta$ occur, as in

$$\int_0^{\frac{1}{2}\pi} \sin^n\theta \cos^m\theta \log(2\sin\theta)\,d\theta, \quad\text{.....................(26)}$$

where n and m are even, we may express $\cos^m\theta$ by means of $\sin\theta$, and so reduce (26) to integrals of the form (24). The particular case where $m=n$ is worthy of notice. Here

$$\int_0^{\frac{1}{2}\pi} \sin^n\theta\cos^n\theta\log(2\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi}\sin^n\theta\cos^n\theta\log(2\cos\theta)\,d\theta$$
$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\frac{\sin^n 2\theta}{2^n}\log(2\sin 2\theta)\,d\theta = \frac{h_n}{2^{n+1}}. \quad\text{...............(27)}$$

A comparison of the two treatments gives a relation between the integrals h. Thus, if $n=4$,

$$h_4 - 2h_6 + h_8 = h_4/2^5.$$

We now proceed to the calculation of the left-hand member of (17) with $\Psi = (b^2-y^2)^{-\frac{1}{2}}$, or, as it may be written,

$$\int_{-b}^{+b}\frac{dy}{\sqrt{(b^2-y^2)}}\left[\left(\gamma+\log\frac{ikr}{2}\right)J_0(kr) + \frac{k^2r^2}{2^2} - \frac{k^4r^4}{2^2.4^2}S_2 + \frac{k^6r^6}{2^2.4^2.6^2}S_3 - \ldots\right]. \text{(28)}$$

The leading term has already been found to be

$$\pi\left(\gamma+\log\frac{ikb}{4}\right) \quad\text{.............................(29)}$$

In (28) r is equal to $\pm(y-\eta)$. Taking, as before,

$$y = b\cos\theta, \quad \eta = b\cos\alpha,$$

we have

$$\int_0^\pi d\theta\left[\left\{\gamma+\log\frac{ikb}{4}+\log\pm 2(\cos\theta-\cos\alpha)\right\}J_0\{kb(\cos\theta-\cos\alpha)\}\right.$$
$$\left.+\frac{k^2b^2(\cos\theta-\cos\alpha)^2}{2^2} - \frac{k^4b^4(\cos\theta-\cos\alpha)^4}{2^2.4^2}\cdot\frac{3}{2} + \frac{k^6b^6(\cos\theta-\cos\alpha)^6}{2^2.4^2.6^6}\cdot\frac{11}{6} - \ldots\right].$$
$$\text{...........(30)}$$

As regards the terms which do not involve $\log(\cos\theta-\cos\alpha)$, we have to deal merely with

$$\int_0^\pi(\cos\theta-\cos\alpha)^n\,d\theta, \quad\text{..........................(31)}$$

where n is an even integer, which, on expansion of the binomial and integration by a known formula, becomes

$$\pi\left[\frac{n-1.n-3.n-5\ldots1}{n.n-2.n-4\ldots2} + \frac{n.n-1}{1.2}\,\frac{n-3.n-5\ldots1}{n-2.n-4\ldots2}\cos^2\alpha\right.$$
$$\left.+\frac{n.n-1.n-2.n-3}{1.2.3.4}\,\frac{n-5.n-7\ldots1}{n-4.n-6\ldots2}\cos^4\alpha + \ldots + \cos^n\alpha\right]. \quad\text{...(32)}$$

Thus, if $n=2$, we get $\pi\left[\frac{1}{2}+\cos^2\alpha\right]$. If $n=4$,

$$\pi\left[\frac{3\,.\,1}{4\,.\,2}+\frac{4\,.\,3}{1\,.\,2}\frac{1}{2}\cos^2\alpha+\cos^4\alpha\right],\quad \text{and so on.}$$

The coefficient of (31), or (32), in (30) is

$$(-1)^{\frac{1}{2}n}\frac{k^n b^n}{2^2\,.\,4^2\ldots n^2}\left[\gamma+\log\frac{ikb}{4}-S_{\frac{1}{2}n}\right]. \quad\ldots\ldots\ldots\ldots\ldots(33)$$

At the centre of the aperture where $\eta=0$, $\cos\alpha=0$, (32) reduces to its first term. At the edges where $\cos\alpha=\pm 1$, we may obtain a simpler form directly from (31). Thus

$$(31)=\int_0^\pi(1\pm\cos\theta)^n\,d\theta=2^n\pi\frac{2n-1\,.\,2n-3\ldots 1}{2n\,.\,2n-2\ldots 2}=\pi\frac{2n-1\,.\,2n-3\ldots 1}{n\,.\,n-1\,.\,n-2\ldots 1}. \quad\ldots\ldots\ldots\ldots(34)$$

For example, if $n=6$,

$$(34)=\pi\frac{11\,.\,9\,.\,7\,.\,5\,.\,3\,.\,1}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}=\frac{231\pi}{16}.$$

We have also in (30) to consider (n even)

$$2^{-n}\int_0^\pi d\theta\,(\cos\theta-\cos\alpha)^n\log\{\pm 2(\cos\theta-\cos\alpha)\}$$

$$=\int_0^\alpha d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{4\sin\frac{\theta+\alpha}{2}\sin\frac{\alpha-\theta}{2}\right\}$$

$$+\int_\alpha^\pi d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{4\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}\right\}$$

$$=\int_0^\pi d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{2\sin\frac{\theta+\alpha}{2}\right\}$$

$$+\int_0^\alpha d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{2\sin\frac{\alpha-\theta}{2}\right\}$$

$$+\int_\alpha^\pi d\theta\sin^n\frac{\theta+\alpha}{2}\sin^n\frac{\theta-\alpha}{2}\log\left\{2\sin\frac{\theta-\alpha}{2}\right\}$$

$$=2\int_0^{\frac{1}{2}\pi+\frac{1}{2}\alpha}d\phi\sin^n\phi\sin^n(\phi-\alpha)\log(2\sin\phi)$$

$$+2\int_0^{\frac{1}{2}\pi-\frac{1}{2}\alpha}d\phi\sin^n\phi\sin^n(\phi+\alpha)\log(2\sin\phi)$$

$$=2\int_0^{\frac{1}{2}\pi}d\phi\sin^n\phi\{\sin^n(\phi-\alpha)+\sin^n(\phi+\alpha)\}\log(2\sin\phi)$$

$$+2\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\frac{1}{2}\alpha}d\phi\sin^n\phi\sin^n(\phi-\alpha)\log(2\sin\phi)$$

$$-2\int_{\frac{1}{2}\pi-\frac{1}{2}\alpha}^{\frac{1}{2}\pi}d\phi\sin^n\phi\sin^n(\phi+\alpha)\log(2\sin\phi)$$

$$=2\int_0^{\frac{1}{2}\pi}d\phi\sin^n\phi\{\sin^n(\phi-\alpha)+\sin^n(\phi+\alpha)\}\log(2\sin\phi), \quad\ldots(35)$$

since the last two integrals cancel, as appears when we write $\pi - \psi$ for ϕ, n being even.

In (35)

$$\tfrac{1}{2}\sin^n(\phi+\alpha) + \tfrac{1}{2}\sin^n(\phi-\alpha) = \sin^n\phi\ \cos^n\alpha$$
$$+\frac{n\,.\,n-1}{1\,.\,2}\sin^{n-2}\phi\ \cos^2\phi\ \sin^2\alpha\ \cos^{n-2}\alpha$$
$$+\frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}\sin^{n-4}\phi\ \cos^4\phi\ \sin^4\alpha\ \cos^{n-4}\alpha + \ldots + \cos^n\phi\sin^n\alpha, \quad (36)$$

and thus the result may be expressed by means of the integrals h. Thus if $n=2$,

$$(35) = 4\int_0^{\frac{1}{2}\pi} d\phi\sin^2\phi\,\{\sin^2\phi\cos^2\alpha + \cos^2\phi\sin^2\alpha\}\log(2\sin\phi)$$
$$= 4\,\{(\cos^2\alpha - \sin^2\alpha)\,h_4 + \sin^2\alpha\,h_2\}. \quad \text{.............................(37)}$$

If $n=4$,

$$(35) = 4\int_0^{\frac{1}{2}\pi} d\phi\sin^4\phi\,\{\sin^4\phi\cos^4\alpha + 6\sin^2\phi\cos^2\phi\sin^2\alpha\cos^2\alpha$$
$$+\cos^4\phi\sin^4\alpha\}\log(2\sin\phi)$$
$$= 4\,\{(\cos^4\alpha - 6\sin^2\alpha\cos^2\alpha + \sin^4\alpha)\,h_8$$
$$+(6\sin^2\alpha\cos^2\alpha - 2\sin^4\alpha)\,h_6 + \sin^4\alpha\,h_4\}. \quad \text{...........(38)}$$

If $n=6$,

$$(35) = 4\,\{(\cos^6\alpha - 15\cos^4\alpha\sin^2\alpha + 15\cos^2\alpha\sin^4\alpha - \sin^6\alpha)\,h_{12}$$
$$+(15\cos^4\alpha\sin^2\alpha - 30\cos^2\alpha\sin^4\alpha + 3\sin^6\alpha)\,h_{10}$$
$$+(15\cos^2\alpha\sin^4\alpha - 3\sin^6\alpha)\,h_8 + \sin^6\alpha\,h_6\}. \quad \text{...................(39)}$$

It is worthy of remark that if we neglect the small differences between the h's in (39), it reduces to $4\cos^6\alpha\,h_{12}$, and similarly in other cases.

When n is much higher than 6, the general expressions corresponding to (37), (38), (39) become complicated. If, however, $\cos\alpha$ be either 0, or ± 1, (36) reduces to a single term, viz., $\cos^n\phi$ or $\sin^n\phi$. Thus at the centre ($\cos\alpha = 0$) from either of its forms

$$(35) = 2^{-n}\,.\,2h_n. \quad \text{.................................(40)}$$

On the other hand, at the edges ($\cos\alpha = \pm 1$)

$$(35) = 4\int_0^{\frac{1}{2}\pi} d\phi\sin^{2n}\phi\log(2\sin\phi) = 4h_{2n}. \quad \text{.............(41)}$$

In (30), the object of our quest, the integral (35) occurs with the coefficient

$$(-1)^{\frac{1}{2}n}\frac{2^n k^n b^n}{2^2\,.\,4^2\,.\,6^2\ldots n^2}. \quad \text{.......................(42)}$$

Thus, expanded in powers of kb, (28) or (30) becomes

$$\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\Bigg[\left\{\gamma+\log\frac{ikb}{4}-1\right\}\left\{\tfrac{1}{2}+\cos^2\alpha\right\}$$

$$+\frac{2^3\,.\,2h_4}{\pi}(\cos^2\alpha-\sin^2\alpha)+\frac{2^3\,.\,2h_2}{\pi}\sin^2\alpha\Bigg]$$

$$+\frac{\pi k^4b^4}{2^2\,.\,4^2}\Bigg[\left\{\gamma+\log\frac{ikb}{4}-\frac{3}{2}\right\}\left\{\frac{3}{8}+3\cos^2\alpha+\cos^4\alpha\right\}$$

$$+\frac{2^5\,.\,2h_8}{\pi}(\cos^4\alpha-6\cos^2\alpha\,\sin^2\alpha+\sin^4\alpha)$$

$$+\frac{2^5\,.\,2h_6}{\pi}(6\cos^2\alpha\,\sin^2\alpha-2\sin^4\alpha)+\frac{2^5\,.\,2h_4}{\pi}\sin^4\alpha\Bigg]$$

$$-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\Bigg[\left\{\gamma+\log\frac{ikb}{4}-\frac{11}{6}\right\}\left\{\frac{5}{16}+\frac{45}{8}\cos^2\alpha+\frac{15}{2}\cos^4\alpha+\cos^6\alpha\right\}$$

$$+\frac{2^7\,.\,2h_{12}}{\pi}(\cos^6\alpha-15\cos^4\alpha\,\sin^2\alpha+15\cos^2\alpha\,\sin^4\alpha-\sin^6\alpha)$$

$$+\frac{2^7\,.\,2h_{10}}{\pi}(15\cos^4\alpha\,\sin^2\alpha-30\cos^2\alpha\,\sin^4\alpha+3\sin^6\alpha)$$

$$+\frac{2^7\,.\,2h_8}{\pi}(15\cos^2\alpha\,\sin^4\alpha-3\sin^6\alpha)+\frac{2^7\,.\,2h_6}{\pi}\sin^6\alpha\Bigg]+\ldots \qquad \ldots\ldots(43)$$

At the centre of the aperture ($\cos\alpha=0$), in virtue of (40), a simpler form is available. We have

$$\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\left[\frac{1}{2}\left(\gamma+\log\frac{ikb}{4}-1\right)+\frac{2h_2}{\pi}\right]$$

$$+\frac{\pi k^4b^4}{2^2\,.\,4^2}\left[\frac{3\,.\,1}{4\,.\,2}\left(\gamma+\log\frac{ikb}{4}-\frac{3}{2}\right)+\frac{2h_4}{\pi}\right]$$

$$-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\left[\frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}\left(\gamma+\log\frac{ikb}{4}-\frac{11}{6}\right)+\frac{2h_6}{\pi}\right]$$

$$+\frac{\pi k^8b^8}{2^2\,.\,4^2\,.\,6^2\,.\,8^2}\left[\frac{7\,.\,5\,.\,3\,.\,1}{8\,.\,6\,.\,4\,.\,2}\left(\gamma+\log\frac{ikb}{4}-\frac{25}{12}\right)+\frac{2h_8}{\pi}\right]-\ldots \quad (44)$$

Similarly at the edges, by (34), (41), we have

$$\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\left[\frac{3\,.\,1}{2\,.\,1}\left(\gamma+\log\frac{ikb}{4}-1\right)+2^3\frac{2h_4}{\pi}\right]$$

$$+\frac{\pi k^4b^4}{2^2\,.\,4^2}\left[\frac{7\,.\,5\,.\,3\,.\,1}{4\,.\,3\,.\,2\,.\,1}\left(\gamma+\log\frac{ikb}{4}-\frac{3}{2}\right)+2^5\,\frac{2h_8}{\pi}\right]$$

$$-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\left[\frac{11\,.\,9\,.\,7\,.\,5\,.\,3\,.\,1}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}\left(\gamma+\log\frac{ikb}{4}-\frac{11}{6}\right)+2^7\,\frac{2h_{12}}{\pi}\right]+\ldots \quad (45)$$

For the general value of α, (43) is perhaps best expressed in terms of $\cos\alpha$, equal to η/b. With introduction of the values of h, we have

$$\pi\left(\gamma+\log\frac{ikb}{4}\right)-\frac{\pi k^2b^2}{2^2}\left[\left(\gamma+\log\frac{ikb}{4}\right)\left(\cos^2\alpha+\frac{1}{2}\right)+\frac{1}{2}\cos^2\alpha-\frac{1}{4}\right]$$

$$+\frac{\pi k^4b^4}{2^2\,.\,4^2}\left[\left(\gamma+\log\frac{ikb}{4}\right)\left(\cos^4\alpha+3\cos^2\alpha+\frac{3}{8}\right)+\frac{7}{12}\cos^4\alpha-\frac{5}{4}\cos^2\alpha-\frac{11}{32}\right]$$

$$-\frac{\pi k^6b^6}{2^2\,.\,4^2\,.\,6^2}\left[\left(\gamma+\log\frac{ikb}{4}\right)\left(\cos^6\alpha+\frac{15}{2}\cos^4\alpha+\frac{45}{8}\cos^2\alpha+\frac{5}{16}\right)\right.$$

$$\left.+\frac{37}{60}\cos^6\alpha-\frac{23}{8}\cos^4\alpha-\frac{159}{32}\cos^2\alpha-\frac{73}{192}\right]+\ldots \quad \text{.....................}(46)$$

These expressions are the values of

$$\int_{-b}^{+b}\frac{D(kr)\,dy}{\sqrt{(b^2-y^2)}}, \quad \text{............................}(47)$$

for the various values of η.

We now suppose that $kb=1$. The values for other particular cases, such as $kb=\frac{1}{2}$, may then easily be deduced. For $\cos\alpha=0$, from (44) we have

$$\pi\left(\gamma+\log\frac{i}{4}\right)\left[1-\frac{1}{2^2}\frac{1}{2}+\frac{1}{2^2\,.\,4^2}\frac{3\,.\,1}{4\,.\,2}-\frac{1}{2^2\,.\,4^2\,.\,6^2}\frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}+\ldots\right]$$

$$+\pi\left[\frac{1}{2^2}\frac{1}{4}-\frac{1}{2^2\,.\,4^2}\frac{11}{32}+\frac{1}{2^2\,.\,4^2\,.\,6^2}\frac{73}{192}-\ldots\right]$$

$$=\pi\left(\gamma+\log\frac{i}{4}\right)[1-0{\cdot}12500+0{\cdot}00586+0{\cdot}00013]$$

$$+\pi[0{\cdot}06250-0{\cdot}00537+0{\cdot}00016]$$

$$=\pi\left(\gamma+\log\frac{i}{4}\right)\times 0{\cdot}88073+\pi\times 0{\cdot}05729$$

$$=\pi[-0{\cdot}65528+1{\cdot}3834\,i], \quad \text{....................................}(48)$$

since $\gamma=0{\cdot}577215,\qquad \log 2=0{\cdot}693147,\qquad \log i=\frac{1}{2}\pi i.$

In like manner, if $kb=\frac{1}{2}$, we get still with $\cos\alpha=0$,

$$\pi\left(\gamma+\log\frac{i}{8}\right)[1-0{\cdot}03125+0{\cdot}00037]+\pi[0{\cdot}01562-0{\cdot}00033]$$

$$=\pi[-1{\cdot}4405+1{\cdot}5223\,i] \quad \text{...........}(49)$$

If $kb=2$, we have

$$\pi\left(\gamma+\log\frac{i}{2}\right)[1-0{\cdot}5+0{\cdot}0938-0{\cdot}0087+0{\cdot}0005]$$

$$+\pi[0{\cdot}25-0{\cdot}0859+0{\cdot}0102-0{\cdot}0006]$$

$$=\pi[+0{\cdot}1058+0{\cdot}9199\,i]. \quad \text{..........................}(50)$$

If $kb = 1$ and $\cos\alpha = \pm 1$, we have from (45)

$$\pi\left(\gamma + \log\frac{i}{4}\right)\left[1 - \frac{1}{2^2}\frac{3}{2} + \frac{1}{2^2 . 4^2}\frac{35}{8} - \frac{1}{2^2 . 4^2 . 6^2}\frac{231}{16} + \frac{1}{2^2 . 4^2 . 6^2 . 8^2}\frac{6435}{128} - \frac{1}{2^2 . 4^2 . 6^2 . 8^2 . 10^2}\frac{19 . 17 . 6435}{10 . 9 . 128} + \ldots\right]$$

$$- \pi\left[\frac{1}{2^2}\frac{1}{4} + \frac{1}{2^2 . 4^2}\frac{97}{96} - \frac{1}{2^2 . 4^2 . 6^2}\frac{7303}{960} + \frac{38\cdot084}{2^2 . 4^2 . 6^2 . 8^2} - \frac{170\cdot64}{2^2 . 4^2 . 6^2 . 8^2 . 10^2} + \ldots\right]$$

$$= \pi\left(\gamma + \log\frac{i}{4}\right)[1 - 0\cdot375 + 0\cdot068359 - 0\cdot006266 + 0\cdot000341 - 0\cdot000012]$$

$$- \pi[0\cdot0625 + 0\cdot015788 - 0\cdot003302 + 0\cdot000258 + 0\cdot000012]$$

$$= \pi[-0\cdot63141 + 1\cdot0798\, i]. \qquad \text{(51)}$$

Similarly, if $kb = \frac{1}{2}$, we have

$$\pi\left(\gamma + \log\frac{i}{8}\right)[1 - 0\cdot09375 + 0\cdot00427 - 0\cdot00010]$$

$$- \pi[0\cdot01562 + 0\cdot00099 - 0\cdot00005]$$

$$= \pi[-1\cdot3842 + 1\cdot4301\, i]. \qquad \text{(52)}$$

And if $kb = 2$, with diminished accuracy,

$$\pi\left(\gamma + \log\frac{i}{2}\right)[1 - 1\cdot5 + 1\cdot094 - 0\cdot401 + 0\cdot087 - 0\cdot012 + 0\cdot001]$$

$$- \pi[0\cdot25 + 0\cdot253 - 0\cdot211 + 0\cdot066 - 0\cdot012 + 0\cdot001]$$

$$= \pi[-0\cdot378 + 0\cdot422\, i]. \qquad \text{(53)}$$

As an intermediate value of α we will select $\cos^2\alpha = \frac{1}{2}$. For $kb = 1$, from (46)

$$\pi\left(\gamma + \log\frac{i}{4}\right)[1 - 0\cdot25 + 0\cdot03320 - 0\cdot00222 + \ldots]$$

$$+ \pi[0 - 0\cdot01286 + 0\cdot001522 + \ldots]$$

$$= \pi[-0\cdot6432 + 1\cdot2268\, i]. \qquad \text{(54)}$$

Also, when $kb = \frac{1}{2}$,

$$\pi[-1\cdot4123 + 1\cdot4759\, i]. \qquad \text{(55)}$$

When $kb = 2$, only a rough value is afforded by (46), viz.,

$$\pi[-0\cdot16 + 0\cdot61\, i]. \qquad \text{(56)}$$

The accompanying table exhibits the various numerical results, the factor π being omitted.

TABLE II.

	$kb = \frac{1}{2}$	$kb = 1$	$kb = 2$
$\cos\alpha = 0$	$-1\cdot4405 + 1\cdot5223\, i$	$-0\cdot65528 + 1\cdot3834\, i$	$+0\cdot1058 + 0\cdot9199\, i$
$\cos^2\alpha = \frac{1}{2}$	$-1\cdot4123 + 1\cdot4759\, i$	$-0\cdot6432 + 1\cdot2268\, i$	$-0\cdot16 + 0\cdot61\, i$
$\cos^2\alpha = 1$	$-1\cdot3842 + 1\cdot4301\, i$	$-0\cdot63141 + 1\cdot0798\, i$	$-0\cdot378 + 0\cdot422\, i$

As we have seen already, the tabulated quantity when kb is very small takes the form $\gamma + \log(ikb/4)$, or $\log kb - 0{\cdot}8091 + 1{\cdot}5708i$, whatever may be the value of α. In this case the condition (17) can be completely satisfied with $\Psi = A(b^2 - y^2)^{-\frac{1}{2}}$, A being chosen suitably. When kb is finite, (17) can no longer be satisfied for all values of α. But when $kb = \frac{1}{2}$, or even when $kb = 1$, the tabulated number does not vary greatly with α and we may consider (17) to be approximately satisfied if we make in the first case

$$\pi(-1{\cdot}4123 + 1{\cdot}4759\,i)\,A = -1, \quad \text{.................(57)}$$

and in the second,

$$\pi(-0{\cdot}6432 + 1{\cdot}2268\,i)\,A = -1. \quad \text{.................(58)}$$

The value of ψ, applicable to a point at a distance directly in front of the aperture, is then, as in (16),

$$\psi = -\pi A\left(\frac{\pi}{2\,ikr}\right)^{\frac{1}{2}} e^{-ikr}. \quad \text{........................(59)}$$

In order to obtain a better approximation we require the aid of a second solution with a different form of Ψ. When this is introduced, as an addition to the first solution and again with an arbitrary constant multiplier, it will enable us to satisfy (17) for two distinct values of α, that is of η, and thus with tolerable accuracy over the whole range from $\cos\alpha = 0$ to $\cos\alpha = \pm 1$. Theoretically, of course, the process could be carried further so as to satisfy (17) for any number of assigned values of $\cos\alpha$.

As the second solution we will take simply $\Psi = 1$, so that the left-hand member of (17) is

$$\int_0^{b+\eta} D(kr)\,dr + \int_0^{b-\eta} D(kr)\,dr. \quad \text{....................(60)}$$

If we omit k, which may always be restored by consideration of homogeneity, we have

$$\begin{aligned}(60) = \left(\gamma + \log\frac{i}{2}\right)&\left[b + \eta - \frac{(b+\eta)^3}{2^2\,.\,3} + \frac{(b+\eta)^5}{2^2\,.\,4^2\,.\,5} - \dots\right] \\ &+ \frac{(b+\eta)^3}{2^2\,.\,3} - \frac{(b+\eta)^5}{2^2\,.\,4^2\,.\,5}S_2 + \frac{(b+\eta)^7}{2^2\,.\,4^2\,.\,6^2\,.\,7}S_3 - \dots \\ &+ (b+\eta)\{\log(b+\eta) - 1\} - \frac{(b+\eta)^3}{2^2\,.\,3}\left\{\log(b+\eta) - \frac{1}{3}\right\} \\ &+ \frac{(b+\eta)^5}{2^2\,.\,4^2\,.\,5}\left\{\log(b+\eta) - \frac{1}{5}\right\} - \dots \\ &+ \text{the same expression with the sign of } \eta \text{ changed.}\end{aligned}$$

The leading term in (60) is thus

$$2b(\gamma - 1 + \log\tfrac{1}{2}i) + (b+\eta)\log(b+\eta) + (b-\eta)\log(b-\eta). \quad \text{....(61)}$$

At the centre of the aperture ($\eta = 0$),

$$(61) = 2b\,\{\gamma - 1 + \log \tfrac{1}{2} ib\},$$

and at the edges ($\eta = \pm\, b$),

$$(61) = 2b\,\{\gamma - 1 + \log ib\}.$$

It may be remarked that in (61), the real part varies with η, although the imaginary part is independent of that variable.

The complete expression (60) naturally assumes specially simple forms at the centre and edges of the aperture. Thus, when $\eta = 0$,

$$(60) \div 2b = \left(\gamma + \log \frac{ib}{2}\right)\left[1 - \frac{b^2}{2^2 . 3} + \frac{b^4}{2^2 . 4^2 . 5} - \ldots\right]$$

$$- 1 + \frac{b^2}{2^2 . 3}\left(1 + \frac{1}{3}\right) - \frac{b^4}{2^2 . 4^2 . 5}\left(1 + \frac{1}{2} + \frac{1}{5}\right) + \frac{b^6}{2^2 . 4^2 . 6^2 . 7}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7}\right) - \ldots; \quad \ldots\ldots(62)$$

and, similarly, when $\eta = \pm\, b$,

$$(60) \div 2b = (\gamma + \log ib)\left[1 - \frac{(2b)^2}{2^2 . 3} + \frac{(2b)^4}{2^2 . 4^2 . 5} - \ldots\right].$$

$$-1 + \frac{(2b)^2}{2^2 . 3}\left(1 + \frac{1}{3}\right) - \frac{(2b)^4}{2^2 . 4^2 . 5}\left(1 + \frac{1}{2} + \frac{1}{5}\right) + \frac{(2b)^6}{2^2 . 4^2 . 6^2 . 7}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7}\right) - \ldots. \quad \ldots\ldots(63)$$

To restore k we have merely to write kb for b in the *right-hand members* of (62), (63).

The calculation is straightforward. For the same values as before of kb and of $\cos^2 \alpha$, equal to η^2/b^2, we get for $(60) \div 2b$

TABLE III.

η^2/b^2	$kb = \frac{1}{2}$	$kb = 1$	$kb = 2$
0	$-1{\cdot}7649 + 1{\cdot}5384\, i$	$-1{\cdot}0007 + 1{\cdot}4447\, i$	$-0{\cdot}2167 + 1{\cdot}1198\, i$
$\frac{1}{2}$	$-1{\cdot}4510 + 1{\cdot}4912\, i$	$-0{\cdot}6740 + 1{\cdot}2771\, i$	$-0{\cdot}1079 + 0{\cdot}7166\, i$
1	$-1{\cdot}0007 + 1{\cdot}4447\, i$	$-0{\cdot}2217 + 1{\cdot}1198\, i$	$+0{\cdot}1394 + 0{\cdot}4024\, i$

We now proceed to combine the two solutions, so as to secure a better satisfaction of (17) over the width of the aperture. For this purpose we determine A and B in

$$\Psi = A\,(b^2 - y^2)^{-\frac{1}{2}} + B, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(64)$$

so that (17) may be exactly satisfied at the centre and edges ($\eta = 0$, $\eta = \pm\, b$). The departure from (17) when $\eta^2/b^2 = \frac{1}{2}$ can then be found. If for any value of kb and $\eta = 0$ the first tabular (complex) number is p and the second q, and for $\eta = \pm\, b$ the first is r and the second s, the equations of condition from (17) are

$$\pi A\,.\,p + 2bB\,.\,q = -1, \qquad \pi A\,.\,r + 2bB\,.\,s = -1. \quad \ldots\ldots\ldots(65)$$

When A and B are found, we have in (16)

$$\int_{-b}^{+b} \Psi\, dy = \pi A + 2bB.$$

From (65) we get

$$\pi A = \frac{q-s}{ps-qr}, \qquad 2bB = \frac{r-p}{ps-qr}, \quad \text{.................(66)}$$

so that

$$\int_{-b}^{+b} \Psi\, dy = \frac{q+r-s-p}{ps-qr}. \quad \text{........................(67)}$$

Thus for $kb = 1$ we have

$$p = -0{\cdot}65528 + 1{\cdot}3834\,i, \qquad q = -1{\cdot}0007 + 1{\cdot}4447\,i,$$

$$r = -0{\cdot}63141 + 1{\cdot}0798\,i, \qquad s = -0{\cdot}2217 + 1{\cdot}1198\,i,$$

whence

$$\pi A = +0{\cdot}60008 + 0{\cdot}51828\,i, \qquad 2bB = -0{\cdot}2652 + 0{\cdot}1073\,i,$$

and

$$(67) = +0{\cdot}3349 + 0{\cdot}6256\,i.$$

The above values of πA and $2bB$ are derived according to (17) from the values at the centre and edges of the aperture. The success of the method may be judged by substitution of the values for $\eta^2/b^2 = \frac{1}{2}$. Using these in (17) we get $-0{\cdot}9801 - 0{\cdot}0082\,i$, for what should be -1, a very fair approximation.

In like manner, for $kb = 2$

$$(67) = +0{\cdot}259 + 1{\cdot}2415\,i;$$

and for $kb = \frac{1}{2}$

$$(67) = +0{\cdot}3378 + 0{\cdot}3526\,i.$$

As appears from (16), when k is given, the modulus of (67) may be taken to represent the amplitude of disturbance at a distant point immediately in front, and it is this with which we are mainly concerned. The following table gives the values of Mod. and Mod.^{2} for several values of kb. The first three have been calculated from the simple formula, see (20).

TABLE IV.

kb	Mod.^{2}	Mod.
0·01	0·0174	0·1320
0·05	0·0590	0·2429
0·25	0·1372	0·3704
0·50	0·2384	0·4883
1·00	0·5035	0·7096
2·00	1·608	1·268

The results are applicable to the problem of aerial waves, or shallow water waves, transmitted through a slit in a thin fixed wall, and to electric

(luminous) waves transmitted by a similar slit in a thin perfectly opaque screen, provided that the electric vector is *perpendicular* to the length of the slit.

In curve A, fig. 1, the value of the modulus from the third column of Table IV is plotted against kb.

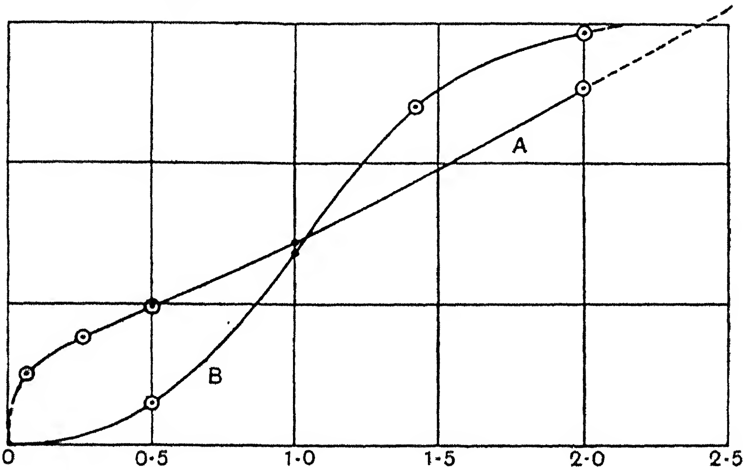


Fig. 1.

When kb is large, the limiting form of (67) may be deduced from a formula, analogous to (12), connecting Ψ and $d\phi/dn$. As in (11),

$$\frac{d\psi}{dx} = \int \Psi \frac{dD}{dx}\, dy,$$

in which, when x is very small, we may take $D = \log r$. Thus

$$\frac{d\psi}{dx} = \Psi \int_{-\infty}^{+\infty} \frac{x\,dy}{x^2 + y^2} = \Psi \left[\tan^{-1}\frac{y}{x}\right]_{-\infty}^{+\infty} = \pi\Psi, \quad \text{or} \quad \Psi = \frac{1}{\pi}\frac{d\psi}{dn}. \;\ldots(68)$$

Now, when kb is large, $d\psi/dn$ tends, except close to the edges, to assume the value ik, and ultimately

$$(67) = \int_{-b}^{+b} \Psi\, dy = \frac{2\,ikb}{\pi}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(69)$$

of which the modulus is $2kb/\pi$ simply, *i.e.* $0{\cdot}637\, kb$.

We now pass on to consider case (ii), where the boundary condition to be satisfied over the wall is $\phi = 0$. Separating from ϕ the solution (χ) which would obtain were the wall unperforated, we have

$$\chi_m = e^{-ikx} - e^{ikx}, \qquad \chi_p = 0, \;\ldots\ldots\ldots\ldots\ldots\ldots(70)$$

giving over the whole plane ($x = 0$),

$$\chi_m = 0, \qquad \chi_p = 0, \qquad d\chi_m/dx = -\,2ik, \qquad d\chi_p/dx = 0.$$

The supplementary solutions ψ, equal to $\phi - \chi$, may be written

$$\psi_m = \int \frac{dD}{dx} \Psi_m dy, \qquad \psi_p = \int \frac{dD}{dx} \Psi_p dy, \quad \text{.............. (71)}$$

where Ψ_m, Ψ_p are functions of y, and the integrations are over the aperture. D as a function of r is given by (13), and r, denoting the distance between dy and the point (x, η), at which ψ_m, ψ_p are estimated, is equal to $\sqrt{\{x^2 + (y-\eta)^2\}}$. The form (71) secures that on the *walls* $\psi_m = \psi_p = 0$, so that the condition of evanescence there, already satisfied by χ, is not disturbed. It remains to satisfy over the *aperture*

$$\psi_m = \psi_p, \qquad -2ik + d\psi_m/dx = d\psi_p/dx. \quad \text{........... (72)}$$

The first of these is satisfied if $\Psi_m = -\Psi_p$, so that ψ_m and ψ_p are equal at any pair of corresponding points on the two sides. The values of $d\psi_m/dx$, $d\psi_p/dx$ are then opposite, and the remaining condition is also satisfied if

$$d\psi_m/dx = ik, \qquad d\psi_p/dx = -ik. \quad \text{................. (73)}$$

At a distance, and if the slit is very narrow, dD/dx may be removed from under the integral sign, so that

$$\psi_p = \frac{dD}{dx} \int_{-b}^{+b} \Psi_p dy, \quad \text{......................... (74)}$$

in which

$$\frac{dD}{dx} = \frac{ikx}{r} \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}. \quad \text{......................... (75)}$$

And even if kb be not small, (74) remains applicable if the distant point be directly in front of the slit, so that $x = r$. For such a point

$$\psi_p = ik \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr} \int_{-b}^{+b} \Psi_p dy. \quad \text{................. (76)}$$

There is a simple relation, analogous to (68), between the value of Ψ_p at any point (η) of the aperture and that of ψ_p at the same point. For in the application of (71) only those elements of the integral contribute which lie infinitely near the point where ψ_p is to be estimated, and for these $dD/dx = x/r^2$. The evaluation is effected by considering in the first instance a point for which x is finite and afterwards passing to the limit. Thus

$$\psi_p = \Psi_p \int \frac{x\,dy}{x^2 + (y-\eta)^2} = \pi \Psi_p. \quad \text{................. (77)}$$

It remains to find, if possible, a form for Ψ_p, or ψ_p, which shall make $d\psi_p/dx$ constant over the aperture, as required by (73). In my former paper, dealing with the case where kb is very small, it was shown that known

theorems relating to the flow of incompressible fluids lead to the desired conclusion. It appeared that (74), (75) give

$$\psi_p = -\frac{k^2 b^2 x}{2r}\left(\frac{\pi}{2\,ikr}\right)^{\frac{1}{2}} e^{-ikr}, \quad \text{.........................(78)}$$

showing that when b is small the transmission falls off greatly, much more than in case (i), see (20). The realised solution from (78) is

$$\psi_p = -\frac{k^2 b^2 x}{2r}\left(\frac{\pi}{2\,kr}\right)^{\frac{1}{2}} \cos(nt - kr - \tfrac{1}{4}\pi), \quad \text{..............(79)}$$

corresponding to $$\chi_m = 2 \sin nt \sin kx. \quad \text{.........................(80)}$$

The former method arrived at a result by assuming certain hydrodynamical theorems. For the present purpose we have to go further, and it will be appropriate actually to verify the constancy of $d\psi/dx$ over the aperture as resulting from the assumed form of Ψ, when kb is small. In this case we may take $D = \log r$, where $r^2 = x^2 + (y-\eta)^2$. From (71), the suffix p being omitted,

$$\frac{d\psi}{dx} = \int_{-b}^{+b} \frac{d^2 D}{dx^2} \Psi \, dy;$$

and herein $$\frac{d^2 D}{dx^2} = -\frac{d^2 D}{d\eta^2} = -\frac{d^2 D}{dy^2} \ (\eta \text{ const.}).$$

Thus, on integration by parts,

$$\frac{d\psi}{dx} = -\left[\Psi \frac{dD}{dy}\right] + \int_{-b}^{+b} \frac{dD}{dy}\frac{d\Psi}{dy}\, dy. \quad \text{..................(81)}$$

In (81) $$\frac{dD}{dy} = \frac{dD}{dr}\frac{dr}{dy} = \frac{y-\eta}{(y-\eta)^2 + x^2},$$

and so long as η is not equal to $\pm b$, it does not become infinite at the limits $(y = \pm b)$, even though $x = 0$. Thus, if Ψ vanish at the limits, the integrated terms in (81) disappear. We now assume for trial

$$\Psi = \sqrt{(b^2 - y^2)}, \quad \text{............................(82)}$$

which satisfies the last-mentioned condition. Writing

$$y = b\cos\theta, \qquad \eta = b\cos\alpha, \qquad x' = x/b,$$

we have $$-\frac{d\psi}{dx} = \int_0^\pi \frac{(\cos\theta - \cos\alpha)^2 + \cos\alpha(\cos\theta - \cos\alpha)}{(\cos\theta - \cos\alpha)^2 + x'^2}\, d\theta. \quad \text{.........(83)}$$

Of the two parts of the integral on the right in (83) the first yields π when $x' = 0$. For the second we have to consider

$$\int_0^\pi \frac{\cos\theta - \cos\alpha}{(\cos\theta - \cos\alpha)^2 + x'^2}\, d\theta, \quad \text{........................(84)}$$

in which $\cos\theta - \cos\alpha$ passes through zero within the range of integration. It will be shown that (84) vanishes ultimately when $x' = 0$. To this end the range of integration is divided into three parts: from 0 to α_1, where $\alpha_1 < \alpha$, from α_1 to α_2, where $\alpha_2 > \alpha$, and lastly from α_2 to π. In evaluating the first and third parts we may put $x' = 0$ at once. And if $z = \tan\frac{1}{2}\theta$

$$\int\frac{d\theta}{\cos\theta - \cos\alpha} = \frac{1}{\sin\alpha}\int\left\{\frac{dz}{\tan\frac{1}{2}\alpha + z} + \frac{dz}{\tan\frac{1}{2}\alpha - z}\right\}.$$

Sin α being omitted, the first and third parts together are thus

$$\log\frac{z+t}{z-t} + \log\frac{t+t_1}{t-t_1} + \log\frac{t_2-t}{t_2+t},$$

where $t = \tan\frac{1}{2}\alpha$, $t_1 = \tan\frac{1}{2}\alpha_1$, $t_2 = \tan\frac{1}{2}\alpha_2$, and z is to be made infinite.

It appears that the two parts taken together vanish, provided t_1, t_2 are so chosen that $t^2 = t_1 t_2$.

It remains to consider the second part, viz.,

$$\int_{\alpha_1}^{\alpha_2}\frac{d\theta\,(\cos\theta - \cos\alpha)}{(\cos\theta - \cos\alpha)^2 + x'^2}, \quad \text{.......................(85)}$$

in which we may suppose the range of integration $\alpha_2 - \alpha_1$ to be very small. Thus

$$(85) = \int_{\alpha_1}^{\alpha_2}\frac{d\theta\,.\,2\sin\frac{1}{2}(\theta+\alpha)\,\sin\frac{1}{2}(\alpha-\theta)}{4\sin^2\frac{1}{2}(\theta+\alpha)\sin^2\frac{1}{2}(\alpha-\theta) + x'^2}$$

$$= -\frac{1}{2\sin\alpha}\log\frac{\sin^2\alpha\,(\alpha_2-\alpha)^2 + x'^2}{\sin^2\alpha\,(\alpha-\alpha_1)^2 + x'^2},$$

and this also vanishes if $\alpha_2 - \alpha = \alpha - \alpha_1$, a condition consistent with the former to the required approximation. We infer that in (83)

$$-\frac{d\psi}{dx} = \pi, \quad \text{..............................(86)}$$

so that, with the aid of a suitable multiplier, (73) can be satisfied. Thus if $\Psi = A\sqrt{(b^2 - y^2)}$, (73) gives $A = ik/\pi$, and the introduction of this into (74) gives (78). We have now to find what departure from (86) is entailed when kb is no longer very small.

Since, in general,

$$d^2D/dx^2 + d^2D/dy^2 + k^2D = 0,$$

we find, as in (81),

$$-\frac{d\psi}{dx} = k^2\int\Psi D\,dy - \int\frac{d\Psi}{dy}\frac{dD}{dy}\,dy, \quad \text{..................(87)}$$

and for the present Ψ has the value defined in (82). The first term on the right of (87) may be treated in the same way as (28) of the former problem, the difference being that $\sqrt{(b^2 - y^2)}$ occurs now in the numerator instead of

the denominator. In (30) we are to introduce under the integral sign the additional factor $k^2b^2\sin^2\theta$. As regards the second term of (87) we have

$$-\int \frac{d\Psi}{dy}\frac{dD}{dy}dy = \int_{-b}^{+b}\frac{y(y-\eta)\,dy}{\sqrt{(b^2-y^2)}}\frac{1}{r}\frac{dD}{dr},$$

where in $\frac{1}{r}\frac{dD}{dr}$ we are to replace r by $\pm(y-\eta)$. We then assume as before $y = b\cos\theta$, $\eta = b\cos\alpha$, and the same definite integrals h_n suffice; but the calculations are more complicated.

We have seen already that the leading term in (87) is π. For the next term we have

$$D = \gamma + \log\frac{ikr}{2}, \qquad \frac{1}{r}\frac{dD}{dr} = \frac{k^2}{4} - \frac{k^2}{2}\left(\gamma + \log\frac{ikr}{2}\right),$$

and thus

$$-\frac{1}{k^2b^2}\frac{d\psi}{dx} = \frac{\pi}{4}\left(\gamma + \log\frac{ikb}{4} + \frac{1}{2}\right)$$
$$+\int_0^\pi d\theta\,(1 - \tfrac{3}{2}\cos^2\theta + \tfrac{1}{2}\cos\alpha\cos\theta)\log\pm 2(\cos\theta - \cos\alpha). \quad\ldots(88)$$

The latter integral may be transformed into

$$2\int_0^{\frac{1}{2}\pi} d\phi\,\{1 - \tfrac{3}{2}\cos^2(2\phi-\alpha) + \tfrac{1}{2}\cos\alpha\,\cos(2\phi-\alpha)$$
$$+1 - \tfrac{3}{2}\cos^2(2\phi+\alpha) + \tfrac{1}{2}\cos\alpha\,\cos(2\phi+\alpha)\}\log(2\sin\phi),$$

and this by means of the definite integrals h is found to be

$$-\frac{\pi}{8}(1 + 2\sin^2\alpha).$$

To this order of approximation the complete value is

$$-\frac{d\psi}{dx} = \pi + \tfrac{1}{4}\pi k^2b^2(\gamma - \sin^2\alpha + \log\tfrac{1}{4}ikb). \quad\ldots\ldots\ldots\ldots(89)$$

For the next two terms I find

$$+\frac{\pi k^4b^4}{512}[(1 + 4\cos^2\alpha)(1 - 4\gamma - 4\log\tfrac{1}{4}ikb)$$
$$+3\sin^4\alpha + \tfrac{1}{3}\cos^4\alpha + 6\sin^2\alpha\cos^2\alpha]$$
$$+\frac{\pi k^6b^6}{2^2.4^2.6}\left[(\tfrac{1}{16} + \tfrac{3}{4}\cos^2\alpha + \tfrac{1}{2}\cos^4\alpha)(\gamma + \log\tfrac{1}{4}ikb - \tfrac{3}{3})\right.$$
$$\left.+\frac{157}{8^2.15}\cos^6\alpha - \frac{13}{8^2.3}\cos^4\alpha\sin^2\alpha - \frac{15}{8^2}\cos^2\alpha\sin^4\alpha - \frac{7}{8^2.3}\sin^6\alpha\right]\ldots(90)$$

When $\cos\alpha = 0$, or ± 1, the calculation is simpler. Thus, when $\cos\alpha = 0$,

$$-\frac{1}{\pi}\frac{d\psi}{dx} = 1 + \frac{k^2b^2}{4}\left(\gamma + \log\frac{ikb}{4} - 1\right) - \frac{k^4b^4}{128}\left(\gamma + \log\frac{ikb}{4} - 1\right)$$
$$+\frac{k^6b^6}{6.4^5}\left(\gamma + \log\frac{ikb}{4} - \frac{5}{4}\right) - \frac{5k^8b^8}{9.4^9}\left(\gamma + \log\frac{ikb}{4} - \frac{22}{15}\right); \quad\ldots\ldots\ldots\ldots(91)$$

and when $\cos\alpha = \pm 1$,

$$-\frac{1}{\pi}\frac{d\psi}{dx} = 1 + \frac{k^2b^2}{4}\left(\gamma + \log\frac{ikb}{4}\right)$$
$$-\frac{k^4b^4}{512}\left\{20\left(\gamma + \log\frac{ikb}{4}\right) - \frac{16}{3}\right\} + \frac{k^6b^6}{6\,.\,4^5}\left\{21\left(\gamma + \log\frac{ikb}{4}\right) - \frac{683}{60}\right\}$$
$$-\frac{k^8b^8}{9\,.\,4^9}\left\{429\left(\gamma + \log\frac{ikb}{4}\right) - 329\right\}, \quad \text{......................}(92)$$

the last term, deduced from h_{14}, h_{16}, being approximate.

For the values of $-\pi^{-1}\,d\psi/dx$ we find from (91), (90), (92) for $kb = \frac{1}{2}$, 1, $\sqrt{2}$, 2:

TABLE V.

	$kb=\frac{1}{2}$	$kb=1$	$kb=\sqrt{2}$	$kb=2$
$\cos\alpha = 0$	$0{\cdot}8448 + 0{\cdot}0974\,i$	$0{\cdot}5615 + 0{\cdot}3807\,i$	$0{\cdot}3123 + 0{\cdot}7383\,i$	$0{\cdot}0102 + 1{\cdot}3899\,i$
$\cos^2\alpha = \frac{1}{2}$	$0{\cdot}8778 + 0{\cdot}0958\,i$	$0{\cdot}6998 + 0{\cdot}3583\,i$	—	$0{\cdot}518 + 1{\cdot}129\,i$
$\cos^2\alpha = 1$	$0{\cdot}9103 + 0{\cdot}0944\,i$	$0{\cdot}8353 + 0{\cdot}3364\,i$	$0{\cdot}8587 + 0{\cdot}5783\,i$	$1{\cdot}020 + 0{\cdot}861\,i$

These numbers correspond to the value of Ψ expressed in (82).

We have now, in pursuance of our method, to seek a second solution with another form of Ψ. The first which suggests itself with $\Psi = 1$ does not answer the purpose. For (81) then gives as the leading term

$$-\frac{d\psi}{dx} = \left[\frac{y-\eta}{(y-\eta)^2 + x^2}\right]_{-b}^{b} = \frac{2b}{b^2 - \eta^2}, \quad \text{.................}(93)$$

becoming infinite when $\eta = \pm b$.

A like objection is encountered if $\Psi = b^2 - y^2$. In this case

$$-\frac{d\psi}{dx} = 2\int\{(y-\eta) + \eta\}\frac{(y-\eta)\,dy}{(y-\eta)^2 + x^2}.$$

The first part gives $4b$ simply when x becomes zero. And

$$2\int\frac{(y-\eta)\,dy}{(y-\eta)^2 + x^2} = \log\frac{(b-\eta)^2 + x^2}{(b+\eta)^2 + x^2};$$

so that
$$-\frac{d\psi}{dx} = 4b + 2\eta\log\frac{b-\eta}{b+\eta}, \quad \text{......................}(94)$$

becoming infinite when $\eta = \pm b$.

So far as this difficulty is concerned we might take $\Psi = (b^2 - y^2)^2$, but another form seems preferable, that is

$$\Psi = b^{-2}(b^2 - y^2)^{3/2}. \quad \text{.........................}(95)$$

With the same notation as was employed in the treatment of (82) we have

$$-\frac{d\psi}{dx} = 3\int_0^\pi \frac{\cos\theta\,(\cos\theta - \cos\alpha)\,d\theta}{(\cos\theta - \cos\alpha)^2 + x'^2} - 3\int_0^\pi \frac{\cos^3\theta\,(\cos\theta - \cos\alpha)}{(\cos\theta - \cos\alpha)^2 + x'^2}\,d\theta.$$

The first of these integrals is that already considered in (83). It yields 3π. In the second integral we replace $\cos^3\theta$ by $\{(\cos\theta - \cos\alpha) + \cos\alpha\}^3$, and we find, much as before, that when $x' = 0$

$$\int_0^\pi \frac{\cos^3\theta\,(\cos\theta - \cos\alpha)\,d\theta}{(\cos\theta - \cos\alpha)^2 + x'^2} = \pi\,(\tfrac{1}{2} + \cos^2\alpha). \quad \text{............(96)}$$

Thus altogether for the leading term we get

$$-\frac{d\psi}{dx} = 3\pi\,(\tfrac{1}{2} - \cos^2\alpha) = 3\pi\,(\tfrac{1}{2} - \eta^2/b^2). \quad \text{..............(97)}$$

This is the complete solution for a fluid regarded as incompressible. We have now to pursue the approximation, using a more accurate value of D than that ($\log r$) hitherto employed.

In calculating the next term, we have the same values of D and $r^{-1}dD/dr$ as for (88); and in place of that equation we now have

$$-\frac{1}{k^2b^2}\frac{d\psi}{dx} = \frac{3\pi}{16}\left(\gamma + \log\frac{ikb}{4} + \frac{1}{2}\right)$$

$$+\int_0^\pi d\theta\,[\tfrac{5}{2}\sin^4\theta - \tfrac{3}{2}\sin^2\theta + \tfrac{3}{2}\sin^2\theta\,\cos\theta\,\cos\alpha]\log\{\pm 2(\cos\theta - \cos\alpha)\}. \quad (98)$$

The integral may be transformed as before, and it becomes

$$\begin{aligned}4\int_0^{\frac{1}{2}\pi} d\phi\,\log(2\sin\phi)\,[\tfrac{5}{2}(&\sin^4 2\phi\,\cos^4\alpha + 6\sin^2 2\phi\,\cos^2 2\phi\,\sin^2\alpha\,\cos^2\alpha \\ &+ \cos^4 2\phi\,\sin^4\alpha) - \tfrac{3}{2}(\sin^2 2\phi\,\cos^2\alpha + \cos^2 2\phi\,\sin^2\alpha) \\ &+ \tfrac{3}{2}\cos\alpha\,\cos 2\phi\,\{\sin^2\alpha\,\cos\alpha + \sin^2 2\phi\,(\cos^3\alpha - 3\sin^2\alpha\,\cos\alpha)\}]. \quad (99)\end{aligned}$$

The evaluation could be effected by expressing the square bracket in terms of powers of $\sin^2\phi$, but it may be much facilitated by use of two lemmas.

If $f(\sin 2\phi, \cos^2 2\phi)$ denote an integral function of $\sin 2\phi$, $\cos^2 2\phi$,

$$\int_0^{\frac{1}{2}\pi} d\phi\,\log(2\sin\phi)\,f(\sin 2\phi, \cos^2 2\phi) = \int_0^{\frac{1}{2}\pi} d\phi\,\log(2\cos\phi)\,f(\sin 2\phi, \cos^2 2\phi)$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} d\phi\,\log(2\sin 2\phi)\,f(\sin 2\phi, \cos^2 2\phi) = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} d\phi\,\log(2\sin\phi)\,f(\sin\phi, \cos^2\phi),$$

$$\text{..................(100)}$$

in which the doubled angles are got rid of.

Again, if m be integral,

$$\int_0^{\frac{1}{2}\pi} d\phi \sin^{2m} 2\phi \cos 2\phi \log(2 \sin \phi)$$

$$= \frac{1}{4m+2}\int \log(2\sin\phi)\, d\sin^{2m+1} 2\phi$$

$$= -\frac{1}{4m+2}\int_0^{\frac{1}{2}\pi} \sin^{2m} 2\phi\,(1+\cos 2\phi)\, d\phi$$

$$= -\frac{1}{4m+2}\int_0^{\frac{1}{2}\pi} \sin^{2m} 2\phi\, d\phi = -\frac{1}{4m+2}\int_0^{\frac{1}{2}\pi} \sin^{2m}\phi\, d\phi$$

$$= -\frac{1}{4m+2}\,\frac{2m-1\,.\,2m-3\ldots 1}{2m\,.\,2m-2\ldots 2}\,\frac{\pi}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(101)$$

For example, if $m = 0$,

$$\int_0^{\frac{1}{2}\pi} d\phi \cos 2\phi \log(2\sin\phi) = -\frac{\pi}{4}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(102)$$

and $(m=1)$

$$\int_0^{\frac{1}{2}\pi} d\phi \sin^2 2\phi \cos 2\phi \log(2\sin\phi) = -\frac{\pi}{24}. \quad\ldots\ldots\ldots\ldots(103)$$

Using these lemmas, we find

$$\begin{aligned}(99) = 5h_4&(\cos^4\alpha - 6\cos^2\alpha\,\sin^2\alpha + \sin^4\alpha)\\ &+ h_2(30\cos^2\alpha\,\sin^2\alpha - 10\sin^4\alpha - 3\cos^2\alpha + 3\sin^2\alpha)\\ &- \tfrac{1}{4}\pi\cos^2\alpha\,(\cos^2\alpha + 3\sin^2\alpha);\end{aligned}$$

and thence, on introduction of the values of h_2, h_4, for the complete value to this order of approximation,

$$\begin{aligned}-\frac{d\psi}{dx} = 3\pi\left(\frac{1}{2} - \cos^2\alpha\right) + \pi k^2 b^2 \Bigg[&\frac{3}{16}\left(\gamma + \frac{1}{2} + \log\frac{ikb}{4}\right)\\ &- \frac{1}{64}(5\cos^4\alpha + 18\cos^2\alpha\sin^2\alpha + 21\sin^4\alpha)\Bigg]. \quad\ldots\ldots(104)\end{aligned}$$

To carry out the calculations to a sufficient approximation with the general value of α would be very tedious. I have limited myself to the extreme cases $\cos\alpha = 0$, $\cos\alpha = \pm 1$. For the former, we have

$$\begin{aligned}-\frac{1}{\pi}\frac{d\psi}{dx} = \frac{3}{2} + \left(\gamma + \log\frac{ikb}{4}\right)&\left\{\frac{3k^2b^2}{16} - \frac{k^4b^4}{256} + \frac{k^6b^6}{2^2\,.\,4^2\,.\,256}\right\}\\ &- \frac{15k^2b^2}{64} + \frac{7k^4b^4}{6\,.\,256} - \frac{11k^6b^6}{4^3\,.\,256\,.\,8}, \quad\ldots\ldots\ldots(105)\end{aligned}$$

and for the latter

$$\begin{aligned}-\frac{1}{\pi}\frac{d\psi}{dx} = -\frac{3}{2} + \left(\gamma + \log\frac{ikb}{4}\right)&\left\{\frac{3k^2b^2}{16} - \frac{7k^4b^4}{16.16} + \frac{33k^6b^6}{4.16.16.16} - \frac{143k^8b^8}{24\,.\,16^4}\right\}\\ &- \frac{5k^2b^2}{64} + \frac{41k^4b^4}{16.64.15} + \frac{1069k^6b^6}{16.3.70.64.64} - \frac{41309k^8b^8}{16^5\,.\,9\,.\,420}\\ &+ \frac{3k^2b^2}{32} + \frac{7k^4b^4}{4.16.16} - \frac{11k^6b^6}{2\,.\,16^3} + \frac{3289k^8b^8}{16^5\,.\,36}. \quad\ldots\ldots\ldots\ldots\ldots(106)\end{aligned}$$

From these formulæ the following numbers have been calculated for the value of $-\pi^{-1}d\psi/dx$:

TABLE VI.

	$kb=\frac{1}{2}$	$kb=1$	$kb=\sqrt{2}$	$kb=2$
$\cos\alpha=0$	$1{\cdot}3716+0{\cdot}0732i$	$1{\cdot}1215+0{\cdot}2885i$	$0{\cdot}8824+0{\cdot}5653i$	$0{\cdot}5499+1{\cdot}0860i$
$\cos\alpha=\pm1$	$-1{\cdot}5634+0{\cdot}0710i$	$-1{\cdot}6072+0{\cdot}2546i$	$-1{\cdot}5693+0{\cdot}4401i$	$-1{\cdot}3952+0{\cdot}6567i$

They correspond to the value of Ψ formulated in (95).

Following the same method as in case (i), we now combine the two solutions, assuming

$$\Psi = A\sqrt{(b^2-y^2)} + Bb^{-2}(b^2-y^2)^{3/2}, \qquad \ldots\ldots\ldots\ldots\ldots(107)$$

and determining A and B so that for $\cos\alpha=0$ and for $\cos\alpha=\pm1$, $d\psi/dx$ shall be equal to $-ik$. The value of ψ at a distance in front is given by (76), in which

$$ik\int\Psi\,dy = \frac{\pi\,.\,ikb^2}{2}\left(A+\frac{3}{4}B\right). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(108)$$

We may take the modulus of (108) as representing the transmitted vibration, in the same way as the modulus of (67) represented the transmitted vibration in case (i).

Using p, q, r, s, as before, to denote the tabulated complex numbers, we have as the equations to determine A and B,

$$Ap+Bq = Ar+Bs = ik/\pi, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(109)$$

so that

$$ik\int\Psi\,dy = -\frac{k^2b^2}{2}\,\frac{s-q+\frac{3}{4}(p-r)}{ps-qr} \qquad \ldots\ldots\ldots\ldots\ldots(110)$$

For the second fraction on the right of (110) and for its modulus we get in the various cases

$kb=\frac{1}{2}$,	$1{\cdot}1470-0{\cdot}1287\,i$,	$1{\cdot}1542$,
$kb=1$,	$1{\cdot}1824-0{\cdot}6986\,i$,	$1{\cdot}3733$,
$kb=\sqrt{2}$,	$0{\cdot}6362-1{\cdot}0258\,i$,	$1{\cdot}2070$,
$kb=2$,	$0{\cdot}1239-0{\cdot}7303\,i$,	$0{\cdot}7407$.

And thence (on introduction of the value of kb) for the modulus of (110) representing the vibration on the same scale as in case (i)

TABLE VII.

kb	Modulus
$\frac{1}{2}$	0·1443
1	0·6866
$\sqrt{2}$	1·2070
2	1·4814

These are the numbers used in the plot of curve B, fig. 1. When kb is much smaller than $\frac{1}{2}$, the modulus may be taken to be $\frac{1}{2}k^2b^2$. When kb is large, the modulus approaches the same limiting form as in case (i).

This curve is applicable to electric, or luminous, vibrations incident upon a thin perfectly conducting screen with a linear perforation when the electric vector is *parallel* to the direction of the slit.

It appears that if the incident light be unpolarised, vibrations perpendicular to the slit preponderate in the transmitted light when the width of the slit is very small, and the more the smaller this width. In the neighbourhood of $kb = 1$, or $2b = \lambda/\pi$, the curves cross, signifying that the transmitted light is unpolarised. When $kb = 1\frac{1}{2}$, or $2b = 3\lambda/2\pi$, the polarisation is reversed, vibrations parallel to the slit having the advantage, but this advantage is not very great. When $kb > 2$, our calculations would hardly succeed, but there seems no reason for supposing that anything distinctive would occur. It follows that if the incident light were white and if the width of the slit were about one-third of the wave-length of yellow-green, there would be distinctly marked opposite polarisations at the ends of the spectrum.

These numbers are in good agreement with the estimates of Fizeau: "Une ligne polarisée perpendiculairement à sa direction a paru être de $\frac{1}{1000}$ de millimètre; une autre, beaucoup moins lumineuse, polarisée parallèlement à sa direction, a été estimée à $\frac{1}{10000}$ de millimètre. Je dois ajouter que ces valeurs ne sont qu'une approximation; elles peuvent être en réalité plus faibles encore, mais il est peu probable qu'elles soient plus fortes. Ce qu'il y a de certain, c'est que la polarisation parallèle n'apparaît que dans les fentes les plus fines, et alors que leur largeur est bien moindre que la longueur d'une ondulation qui est environ de $\frac{1}{2000}$ de millimètre." It will be remembered that the "plane of polarisation" is perpendicular to the electric vector.

It may be well to emphasize that the calculations of this paper relate to an aperture in an *infinitely thin* perfectly conducting screen. We could scarcely be sure beforehand that the conditions are sufficiently satisfied even by a scratch upon a silver deposit. The case of an ordinary spectroscope slit is quite different. It seems that here the polarisation observed with the finest practicable slits corresponds to that from the less fine scratches on silver deposits.

376.

ON THE MOTION OF A VISCOUS FLUID.

[*Philosophical Magazine*, Vol. XXVI. pp. 776—786, 1913.]

It has been proved by Helmholtz* and Korteweg† that when the velocities at the boundary are given, the slow steady motion of an incompressible viscous liquid satisfies the condition of making F, the dissipation, an absolute minimum. If u_0, v_0, w_0 be the velocities in one motion M_0, and u, v, w those of another motion M satisfying the same boundary conditions, the difference of the two u', v', w', where

$$u' = u - u_0, \quad v' = v - v_0, \quad w' = w - w_0, \qquad\qquad (1)$$

will constitute a motion M' such that the boundary velocities vanish. If F_0, F, F' denote the dissipation-functions for the three motions M_0, M, M' respectively, all being of necessity positive, it is shown that

$$F = F_0 + F' - 2\mu \int (u'\nabla^2 u_0 + v'\nabla^2 v_0 + w'\nabla^2 w_0)\, dx\, dy\, dz, \qquad (2)$$

the integration being over the whole volume. Also

$$F' = -\mu \int (u'\nabla^2 u' + v'\nabla^2 v' + w'\nabla^2 w')\, dx\, dy\, dz$$

$$= \mu \int \left[\left(\frac{dw'}{dy} - \frac{dv'}{dz}\right)^2 + \left(\frac{du'}{dz} - \frac{dw'}{dx}\right)^2 + \left(\frac{dv'}{dx} - \frac{du'}{dy}\right)^2\right] dx\, dy\, dz. \qquad (3)$$

These equations are purely kinematical, if we include under that head the incompressibility of the fluid. In the application of them by Helmholtz and Korteweg the motion M_0 is supposed to be that which would be steady if small enough to allow the neglect of the terms involving the second powers of the velocities in the dynamical equations. We then have

$$\mu\nabla^2 (u_0, v_0, w_0) = \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)(V\rho + p_0). \qquad (4)$$

* *Collected Works*, Vol. I. p. 223 (1869).

† *Phil. Mag.* Vol. XVI. p. 112 (1883).

where V is the potential of impressed forces. In virtue of (4)

$$\int (u'\nabla^2 u_0 + v'\nabla^2 v_0 + w'\nabla^2 w_0)\, dx\, dy\, dz = 0, \qquad\qquad (5)$$

if the space occupied by the fluid be simply connected, or in any case if V be single-valued. Hence

$$F = F_0 + F', \qquad\qquad (6)$$

or since F' is necessarily positive, the motion M_0 makes F an absolute minimum. It should be remarked that F' can vanish only for a motion such as can be assumed by a solid body (Stokes), and that such a motion could not make the boundary velocities vanish. The motion M_0 determined by (4) is thus unique.

The conclusion expressed in (6) that M_0 makes F an absolute minimum is not limited to the supposition of a slow motion. All that is required to ensure the fulfilment of (5), on which (6) depends, is that $\nabla^2 u_0$, $\nabla^2 v_0$, $\nabla^2 w_0$ should be the derivatives of some single-valued function. Obviously it would suffice that $\nabla^2 u_0$, $\nabla^2 v_0$, $\nabla^2 w_0$ vanish, as will happen if the motion have a velocity-potential. Stokes* remarked long ago that when there is a velocity-potential, not only are the ordinary equations of fluid motion satisfied, but the equations obtained when friction is taken into account are satisfied likewise. A motion with a velocity-potential can always be found which shall have prescribed *normal* velocities at the boundary, and the tangential velocities are thereby determined. If these agree with the prescribed tangential velocities of a viscous fluid, all the conditions are satisfied by the motion in question. And since this motion makes F an absolute minimum, it cannot differ from the motion determined by (4) with the same boundary conditions. We may arrive at the same conclusion by considering the general equation of motion

$$\rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) = \mu\nabla^2 u - \frac{d(\rho V + p)}{dx}. \qquad\qquad (7)$$

If there be a velocity-potential ϕ, so that $u = d\phi/dx$, &c.,

$$u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} = \frac{1}{2}\frac{d}{dx}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\}; \qquad\qquad (8)$$

and then (7) and its analogues reduce practically to the form (4) if the motion be steady.

Other cases where F is an absolute minimum are worthy of notice. It suffices that

$$\nabla^2 u_0 = \frac{dH}{dx}, \quad \nabla^2 v_0 = \frac{dH}{dy}, \quad \nabla^2 w_0 = \frac{dH}{dz}, \qquad\qquad (9)$$

* *Camb. Trans.* Vol. IX. (1850); *Math. and Phys. Papers*, Vol. III. p. 73.

where H is a single-valued function, subject to $\nabla^2 H = 0$. If ξ_0, η_0, ζ_0 be the rotations,

$$2\nabla^2\xi_0 = \nabla^2\left(\frac{dw_0}{dy} - \frac{dv_0}{dz}\right) = \frac{d}{dy}\nabla^2 w_0 - \frac{d}{dz}\nabla^2 v_0 = 0\,;$$

and thus (9) requires that

$$\nabla^2\xi_0 = 0, \quad \nabla^2\eta_0 = 0, \quad \nabla^2\zeta_0 = 0. \qquad \text{(10)}$$

In two dimensions the dynamical equation reduces to $D\zeta_0/Dt = 0$*, so that ζ_0 is constant along a stream-line. Among the cases included are the motion between two planes

$$u_0 = A + By + Cy^2, \quad v_0 = 0, \quad w_0 = 0, \qquad \text{(11)}$$

and the motion in circles between two coaxal cylinders ($\zeta_0 =$ constant). Also, without regard to the form of the boundary, the uniform rotation, as of a solid body, expressed by

$$u_0 = Cy, \quad v_0 = -Cx. \qquad \text{(12)}$$

In all these cases F is an absolute minimum.

Conversely, if the conditions (9) be not satisfied, it will be possible to find a motion for which $F < F_0$. To see this choose a place as origin of coordinates where $d\nabla^2 u_0/dy$ is not equal to $d\nabla^2 v_0/dx$. Within a small sphere described round this point as centre let $u' = Cy$, $v' = -Cx$, $w' = 0$, and let $u' = 0$, $v' = 0$, $w' = 0$ outside the sphere, thus satisfying the prescribed boundary conditions. Then in (2)

$$\int (u'\nabla^2 u_0 + v'\nabla^2 v_0 + w'\nabla^2 w_0)\, dx\, dy\, dz = C\int (y\nabla^2 u_0 - x\nabla^2 v_0)\, dx\, dy\, dz, \quad \text{...(13)}$$

the integration being over the sphere. Within this small region we may take

$$\nabla^2 u_0 = (\nabla^2 u_0)_0 + \frac{d\nabla^2 u_0}{dx_0}x + \frac{d\nabla^2 u_0}{dy_0}y + \frac{d\nabla^2 u_0}{dz_0}z,$$

$$\nabla^2 v_0 = (\nabla^2 v_0)_0 + \frac{d\nabla^2 v_0}{dx_0}x + \frac{d\nabla^2 v_0}{dy_0}y + \frac{d\nabla^2 v_0}{dz_0}z\,;$$

so that (13) reduces to

$$C\left(\frac{d\nabla^2 u_0}{dy_0} - \frac{d\nabla^2 v_0}{dx_0}\right)\int (y^2 \text{ or } x^2)\, dx\, dy\, dz.$$

Since the sign of C is at disposal, this may be made positive or negative at pleasure. Also F' in (2) may be neglected as of the second order when u', v', w' are small enough. It follows that F is not an absolute minimum for u_0, v_0, w_0, unless the conditions (9) are satisfied.

Korteweg has also shown that the slow motion of a viscous fluid denoted by M_0 is *stable*. "When in a given region occupied by viscous

* Where $D/Dt = d/dt + u\,d/dx + v\,d/dy + w\,d/dz$.

incompressible fluid there exists at a certain moment a mode of motion M which does not satisfy equation (4), then, the velocities along the boundary being maintained constant, the change which must occur in the mode of motion will be such (neglecting squares and products of velocities) that the dissipation of energy by internal friction is constantly decreasing till it reaches the value F_0 and the mode of motion becomes identical with M_0."

This theorem admits of instantaneous proof. If the terms of the second order are omitted, the equations of motion, such as (7), are linear, and any two solutions may be superposed. Consider two solutions, both giving the same velocities at the boundary. Then the difference of these is also a solution representing a possible motion with zero velocities at the boundary. But such a motion necessarily comes to rest. Hence with flux of time the two original motions tend to become and to remain identical. If one of these is the steady motion, the other must tend to become coincident with it.

The stability of the *slow* steady motion of a viscous fluid, or (as we may put it) the steady motion of a *very* viscous fluid, is thus ensured. When the circumstances are such that the terms of the second order must be retained, there is but little definite knowledge as to the character of the motion in respect of stability. Viscous fluid, contained in a vessel which rotates with uniform velocity, would be expected to acquire the same rotation and ultimately to revolve as a solid body, but the expectation is perhaps founded rather upon observation than upon theory. We might, however, argue that any other event would involve perpetual dissipation which could only be met by a driving force applied to the vessel, since the kinetic energy of the motion could not for ever diminish. And such a maintained driving couple would generate angular momentum without limit—a conclusion which could not be admitted. But it may be worth while to examine this case more closely.

We suppose as before that u_0, v_0, w_0 are the velocities in the steady motion M_0 and u, v, w those of the motion M, both motions satisfying the dynamical equations, and giving the prescribed boundary velocities; and we consider the expression for the kinetic energy T' of the motion (1) which is the difference of these two, and so makes the velocities vanish at the boundary. The motion M' with velocities u', v', w' does not in general satisfy the dynamical equations. We have

$$\frac{1}{\rho}\frac{dT'}{dt} = \int\left\{u'\frac{du'}{dt} + v'\frac{dv'}{dt} + w'\frac{dw'}{dt}\right\} dx\,dy\,dz. \quad \text{............(14)}$$

In equations (7) which are satisfied by the motion M we substitute $u = u_0 + u'$, &c.; and since the solution M_0 is steady we have

$$\frac{du_0}{dt} = \frac{dv_0}{dt} = \frac{dw_0}{dt} = 0. \quad \text{..........................(15)}$$

We further suppose that $\nabla^2 u_0$, $\nabla^2 v_0$, $\nabla^2 w_0$ are derivatives of a function H, as in (9). This includes the case of uniform rotation expressed by

$$u_0 = y, \quad v_0 = -x, \quad w_0 = 0, \qquad \text{......................(16)}$$

as well as those where there is a velocity-potential. Thus (7) becomes

$$\frac{du'}{dt} = \nu \nabla^2 u' - \frac{d\varpi}{dx} - (u_0 + u')\left(\frac{du_0}{dx} + \frac{du'}{dx}\right)$$
$$- (v_0 + v')\left(\frac{du_0}{dy} + \frac{du'}{dy}\right) - (w_0 + w')\left(\frac{du_0}{dz} + \frac{du'}{dz}\right), \qquad \text{......(17)}$$

with two analogous equations, where

$$\varpi = V + p/\rho - \nu H, \quad \nu = \mu/\rho. \qquad \text{....................(18)}$$

These values of du'/dt, &c., are to be substituted in (14).

In virtue of the equation of continuity to which u', v', w' are subject, the terms in ϖ contribute nothing to dT'/dt, as appears at once on integration by parts. The remaining terms in dT'/dt are of the first, second, and third degree in u', v', w'. Those of the first degree contribute nothing, since u_0, v_0, w_0 satisfy equations such as

$$u_0 \frac{du_0}{dx} + v_0 \frac{du_0}{dy} + w_0 \frac{du_0}{dz} = - \frac{d\varpi_0}{dx}. \qquad \text{................(19)}$$

The terms of the third degree are

$$-\int \Big[u' \left\{ u' \frac{du'}{dx} + v' \frac{du'}{dy} + w' \frac{du'}{dz} \right\}$$
$$+ v' \left\{ u' \frac{dv'}{dx} + v' \frac{dv'}{dy} + w' \frac{dv'}{dz} \right\}$$
$$+ w' \left\{ u' \frac{dw'}{dx} + v' \frac{dw'}{dy} + w' \frac{dw'}{dz} \right\} \Big] \, dx\, dy\, dz,$$

which may be written

$$-\frac{1}{2} \int \Big[u' \frac{d(u'^2 + v'^2 + w'^2)}{dx} + v' \frac{d(u'^2 + v'^2 + w'^2)}{dy}$$
$$+ w' \frac{d(u'^2 + v'^2 + w'^2)}{dz} \Big] \, dx\, dy\, dz;$$

and this vanishes for the same reason as the terms in ϖ.

We are left with the terms of the second degree in u', v', w'. Of these the part involving ν is

$$\nu \int [u' \nabla^2 u' + v' \nabla^2 v' + w' \nabla^2 w'] \, dx\, dy\, dz. \qquad \text{..............(20)}$$

So far as this part is concerned, we see from (3) that

$$dT'/dt = -F', \qquad \text{............................(21)}$$

F' being the dissipation-function calculated from u', v', w'.

Of the remaining 18 terms of the second degree, 9 vanish as before when integrated, in virtue of the equation of continuity satisfied by u_0, v_0, w_0. Finally we have*

$$\frac{dT'}{dt} = -F' - \rho \int \left[u' \left\{ u' \frac{du_0}{dx} + v' \frac{du_0}{dy} + w' \frac{du_0}{dz} \right\} + v' \left\{ u' \frac{dv_0}{dx} + v' \frac{dv_0}{dy} + w' \frac{dv_0}{dz} \right\} + w' \left\{ u' \frac{dw_0}{dx} + v' \frac{dw_0}{dy} + w' \frac{dw_0}{dz} \right\} \right] dx\, dy\, dz. \quad \text{........(22)}$$

If the motion u_0, v_0, w_0 be in two dimensions, so that $w_0 = 0$, while u and v_0 are independent of z, (22) reduces to

$$\frac{dT'}{dt} = -F' - \rho \int \left[u'^2 \frac{du_0}{dx} + v'^2 \frac{dv_0}{dy} + u'v' \left(\frac{du_0}{dy} + \frac{dv_0}{dx} \right) \right] dx\, dy\, dz. \quad \text{...(23)}$$

Under this head comes the case of uniform rotation expressed in (16), for which

$$\frac{du_0}{dx} = 0, \quad \frac{dv_0}{dy} = 0, \quad \frac{du_0}{dy} + \frac{dv_0}{dx} = 0.$$

Here then $dT'/dt = -F'$ simply, that is T' continually diminishes until it becomes insensible. Any motion superposed upon that of uniform rotation gradually dies out.

When the motion u_0, v_0, w_0 has a velocity-potential ϕ, (22) may be written

$$\frac{dT'}{dt} = -F' - \rho \int \left[u'^2 \frac{d^2\phi}{dx^2} + v'^2 \frac{d^2\phi}{dy^2} + w'^2 \frac{d^2\phi}{dz^2} + 2u'v' \frac{d^2\phi}{dx\, dy} + 2v'w' \frac{d^2\phi}{dy\, dz} + 2w'u' \frac{d^2\phi}{dz\, dx} \right] dx\, dy\, dz. \quad \text{......(24)}$$

So far as I am aware, no case of complete stability for all values of μ is known, other than the motion possible to a solid body above considered. It may be doubted whether such cases exist. Under the head of (24) a simple example occurs when $\phi = \tan^{-1}(y/x)$, the irrotational motion taking place in concentric circles. Here if $r^2 = x^2 + y^2$,

$$\frac{dT'}{dt} = -F' - 2\rho \int \left[\frac{xy}{r^4} (u'^2 - v'^2) + \frac{y^2 - x^2}{r^4} u'v' \right] dx\, dy\, dz. \quad \text{......(25)}$$

* Compare O. Reynolds, *Phil. Trans.* 1895, Part I. p. 146. In Lorentz's deduction of a similar equation (*Abhandlungen*, Vol. I. p. 46) the additional motion is assumed to be small. This memoir, as well as that of Orr referred to below, should be consulted by those interested. See also Lamb's *Hydrodynamics*, § 346.

If the superposed motion also be two-dimensional, it may be expressed by means of a stream-function ψ. We have in terms of polar coordinates

$$u' = \frac{d\psi}{dy} = \frac{d\psi}{dr}\sin\theta + \frac{1}{r}\frac{d\psi}{d\theta}\cos\theta,$$

$$-v' = \frac{d\psi}{dx} = \frac{d\psi}{dr}\cos\theta - \frac{1}{r}\frac{d\psi}{d\theta}\sin\theta,$$

so that

$$u'^2 - v'^2 = (\cos^2\theta - \sin^2\theta)\left\{\frac{1}{r^2}\left(\frac{d\psi}{d\theta}\right)^2 - \left(\frac{d\psi}{dr}\right)^2\right\} + \frac{4\sin\theta\cos\theta}{r}\frac{d\psi}{dr}\frac{d\psi}{d\theta},$$

$$-u'v' = \cos\theta\sin\theta\left\{\left(\frac{d\psi}{dr}\right)^2 - \frac{1}{r^2}\left(\frac{d\psi}{d\theta}\right)^2\right\} + \frac{\cos^2\theta - \sin^2\theta}{r}\frac{d\psi}{dr}\frac{d\psi}{d\theta}.$$

Thus

$$\cos\theta\sin\theta\,(u'^2 - v'^2) - (\cos^2\theta - \sin^2\theta)\,u'v' = \frac{1}{r}\frac{d\psi}{dr}\frac{d\psi}{d\theta}, \quad \text{......(26)}$$

and (25) becomes

$$\frac{dT'}{dt} = -F' - 2\rho\iiint\frac{1}{r^2}\frac{d\psi}{dr}\frac{d\psi}{d\theta}\,dr\,d\theta\,dz, \quad \text{...............(27)}$$

T', F', as well as the last integral, being proportional to z.

We suppose the motion to take place in the space between two coaxal cylinders which revolve with appropriate velocities. If the additional motion be also symmetrical about the axis, the stream-lines are circles, and ψ is a function of r only. The integral in (27) then disappears and dT'/dt reduces to $-F'$, so that under this restriction * the original motion is stable. The experiments of Couette† and of Mallock‡, made with revolving cylinders, appear to show that when u', v', w' are not specially restricted the motion is unstable. It may be of interest to follow a little further the indications of (27).

The general value of ψ is

$$\psi = C_0 + C_1\cos\theta + S_1\sin\theta + \ldots + C_n\cos n\theta + S_n\sin n\theta, \quad \text{......(28)}$$

C_n, S_n being functions of r, whence

$$\int\frac{d\psi}{dr}\frac{d\psi}{d\theta}\,d\theta = \pi\Sigma n\left(S_n\frac{dC_n}{dr} - C_n\frac{dS_n}{dr}\right), \quad \text{..............(29)}$$

n being 1, 2, 3, &c. If S_n, C_n differ only by a constant multiplier, (29) vanishes. This corresponds to

$$\psi = R_0 + R_1\cos(\theta + \epsilon_1) + \ldots + R_n\cos n(\theta + \epsilon_n) + \ldots, \quad \text{......(30)}$$

* We may imagine a number of thin, coaxal, freely rotating cylinders to be interposed between the extreme ones whose motion is prescribed.

† *Ann. d. Chimie*, t. XXI. p. 433 (1890).

‡ *Proc. Roy. Soc.* Vol. LIX. p. 38 (1895).

where R_0, R_1, &c. are functions of r, while ϵ_1, ϵ_2, &c. are constants. If ψ can be thus limited, dT'/dt reduces to $-F'$, and the original motion is stable.

In general
$$\frac{dT'}{dt} = -F' - 2\pi\rho z \int \Sigma n \left(S_n \frac{dC_n}{dr} - C_n \frac{dS_n}{dr}\right) \frac{dr}{r^2}. \qquad (31)$$

C_n, S_n must be such as to give at the boundaries

$$C_n = 0, \quad dC_n/dr = 0, \quad S_n = 0, \quad dS_n/dr = 0; \qquad (32)$$

otherwise they are arbitrary functions of r. It may be noticed that the sign of any term in (29) may be altered at pleasure by interchange of C_n and S_n.

When μ is great, so that the influence of F preponderates, the motion is stable. On the other hand when μ is small, the motion is probably unstable, unless special restrictions can be imposed.

A similar treatment applies to the problem of the uniform shearing motion of a fluid between two parallel plane walls, defined by

$$u_0 = A + By, \quad v_0 = 0, \quad w_0 = 0. \qquad (33)$$

From (23)
$$\frac{dT'}{dt} = -F' - \rho B \iint u' v' \, dx \, dy. \qquad (34)$$

If in the superposed motion $v' = 0$, the double integral vanishes and the original motion is stable. More generally, if the stream-function of the superposed motion be

$$\psi = C \cos kx + S \sin kx, \qquad (35)$$

where C, S are functions of y, we find

$$\frac{dT'}{dt} = -F' + \rho B \iint \frac{d\psi}{dy} \frac{d\psi}{dx} \, dx \, dy$$
$$= -F' + \frac{\rho B \,.\, kx}{2} \int \left(S \frac{dC}{dy} - C \frac{dS}{dy}\right) dy. \qquad (36)$$

Here again if the motion can be such that C and S differ only by a constant multiplier, the integral would vanish. When μ is small and there is no special limitation upon the disturbance, instability probably prevails. The question whether μ is to be considered great or small depends of course upon the other data of the problem. If D be the distance between the planes, we have to deal with BD^2/ν (Reynolds).

In an important paper* Orr, starting from equation (34), has shown that if BD^2/ν is less than 177 "every disturbance must automatically decrease, and that (for a higher value than 177) it is possible to prescribe a disturbance which will increase for a time." We must not infer that when

* *Proc. Roy. Irish Acad.* 1907.

$BD^2/\nu > 177$ the regular motion is necessarily unstable. As the fluid moves under the laws of dynamics, the initial increase of certain disturbances may after a time be exchanged for a decrease, and this decrease may be without limit.

At the other extreme when ν is very small, observation shows that the tangential traction on the walls, moving (say) with velocities $\pm U$, tends to a statistical uniformity and to become proportional, no longer to U, but to U^2. If we assume this law to be absolute in the region of high velocity, the principle of dynamical similarity leads to rather remarkable conclusions. For the tangential traction, having the dimensions of a pressure, must in general be of the form

$$\rho U^2 . f\left(\frac{\nu}{UD}\right), \qquad \text{.............................(37)}$$

D being the distance between the walls, and f an arbitrary function. In the regular motion (z large) $f(z) = 2z$, and (37) is proportional to U. If (37) is proportional to U^2, f must be a constant and the traction becomes independent not only of μ, but *also* of D.

If the velocity be not quite so great as to reduce f to constancy, we may take

$$f(z) = a + bz,$$

where a and b are numerical constants, so that (37) becomes

$$a\rho U^2 + b\mu U/D. \qquad \text{.............................(38)}$$

It could not be assumed without further proof that b has the value (2) appropriate to a large z; nevertheless, Korteweg's equation (6) suggests that such may be the case.

From data given by Couette I calculate that in C.G.S. measure

$$a = \cdot 000027.$$

The tangential traction is thus about a twenty thousandth part of the pressure ($\frac{1}{2}\rho U^2$) due to the normal impact of the fluid moving with velocity U.

Even in cases where the steady motion of a viscous fluid satisfying the dynamical equations is certainly unstable, there is a distinction to be attended to which is not without importance. It may be a question of the *time* during which the fluid remains in an unstable condition. When fluid moves between two coaxal cylinders, the instability has an indefinite time in which to develop itself. But it is otherwise in many important problems. Suppose that fluid has to move through a narrow place, being guided for example by hyperbolic surfaces, either in two dimensions, or in three with symmetry about an axis. If the walls have suitable tangential velocities, the motion

may be irrotational. This irrotational motion is that which would be initiated from rest by propellent impulses acting at a distance. If the viscosity were great, the motion would be steady and stable; if the viscosity is less, it still satisfies the dynamical equations, but is (presumably) unstable. But the instability, as it affects any given portion of the fluid, has a very short duration. Only as it approaches the narrows has the fluid any considerable velocity, and as soon as the narrows are passed the velocity falls off again. Under these circumstances it would seem probable that the instability in the narrows would be of little consequence, and that the irrotational motion would practically hold its own. If this be so, the tangential movement of the walls exercises a profound influence, causing the fluid to follow the walls on the down stream side, instead of shooting onwards as a jet—the behaviour usually observed when fluid is invited to follow fixed divergent walls, unless indeed the expansion is very gradual.

377.

ON THE STABILITY OF THE LAMINAR MOTION OF AN INVISCID FLUID.

[*Philosophical Magazine*, Vol. XXVI. pp. 1001—1010, 1913.]

THE equations of motion of an inviscid fluid are satisfied by a motion such that U, the velocity parallel to x, is an arbitrary function of y only, while the other component velocities V and W vanish. The motion may be supposed to be limited by two fixed plane walls for each of which y has a constant value. In order to investigate the stability of the motion, we superpose upon it a two-dimensional disturbance u, v, where u and v are regarded as small. If the fluid is incompressible,

$$\frac{du}{dx}+\frac{dv}{dy}=0\,; \quad \text{.................................(1)}$$

and if the squares and products of small quantities are neglected, the hydrodynamical equations give*

$$\left(\frac{d}{dt}+U\frac{d}{dx}\right)\left(\frac{du}{dy}-\frac{dv}{dx}\right)+\frac{d^2U}{dy^2}v=0. \quad \text{..................(2)}$$

From (1) and (2), if we assume that as functions of t and x, u and v are proportional to $e^{i(nt+kx)}$, where k is real and n may be real or complex,

$$\left(\frac{n}{k}+U\right)\left(\frac{d^2v}{dy^2}-k^2v\right)+\frac{d^2U}{dy^2}v=0. \quad \text{.....................(3)}$$

In the paper quoted it was shown that under certain conditions n could not be complex; and it may be convenient to repeat the argument. Let

$$n/k=p+iq, \quad v=\alpha+i\beta,$$

* *Proceedings of London Mathematical Society*, Vol. XI. p. 57 (1880); *Scientific Papers*, Vol. I. p. 485. Also Lamb's *Hydrodynamics*, § 345.

where p, q, α, β are real. Substituting in (3) and equating separately to zero the real and imaginary parts, we get

$$\frac{d^2\alpha}{dy^2} = k^2\alpha + \frac{d^2U}{dy^2}\frac{(p+U)\alpha + q\beta}{(p+U)^2+q^2},$$

$$\frac{d^2\beta}{dy^2} = k^2\beta + \frac{d^2U}{dy^2}\frac{-q\alpha + (p+U)\beta}{(p+U)^2+q^2};$$

whence if we multiply the first by β and the second by α and subtract,

$$\frac{d}{dy}\left(\beta\frac{d\alpha}{dy} - \alpha\frac{d\beta}{dy}\right) = \frac{d^2U}{dy^2}\frac{q(\alpha^2+\beta^2)}{(p+U)^2+q^2}. \quad \text{..............(4)}$$

At the limits, corresponding to finite or infinite values of y, we suppose that v, and therefore both α and β, vanish. Hence when (4) is integrated with respect to y between these limits, the left-hand member vanishes and we infer that q also must vanish unless d^2U/dy^2 changes sign. Thus in the motion between walls if the velocity curve, in which U is ordinate and y abscissa, be of one curvature throughout, n must be wholly real; otherwise, so far as this argument shows, n may be complex and the disturbance exponentially unstable.

Two special cases at once suggest themselves. If the motion be that which is possible to a viscous fluid moving steadily between two fixed walls under external pressure or impressed force, so that for example $U = y^2 - b^2$, d^2U/dy^2 is a finite constant, and complex values of n are clearly excluded. In the case of a simple shearing motion, exemplified by $U = y$, $d^2U/dy^2 = 0$, and no inference can be drawn from (4). But referring back to (3), we see that in this case if n be complex,

$$\frac{d^2v}{dy^2} - k^2v = 0 \quad \text{..............................(5)}$$

would have to be satisfied over the whole range between the limits where $v = 0$. Since such satisfaction is not possible, we infer that here too a complex n is excluded.

It may appear at first sight as if real, as well as complex, values of n were excluded by this argument. But if n be such that $n/k + U$ vanishes anywhere within the range, (5) need not there be satisfied. In other words, the arbitrary constants which enter into the solution of (5) may there change values, subject only to the condition of making v continuous. The terminal conditions can then be satisfied. Thus any value of $-n/k$ is admissible which coincides with a value of U to be found within the range. But other real values of n are excluded.

Let us now examine how far the above argument applies to real values of n, when d^2U/dy^2 in (3) does not vanish throughout. It is easy to recognize

that here also any value of $-kU$ is admissible, and for the same reason as before, viz., that when $n+kU=0$, dv/dy may be discontinuous. Suppose, for example, that there is but one place where $n+kU=0$. We may start from either wall with $v=0$ and with an arbitrary value of dv/dy and gradually build up the solutions inwards so as to satisfy (3)*. The process is to be continued on both sides until we come to the place where $n+kU=0$. The two values there found for v and for dv/dy will presumably disagree. But by suitable choice of the relative initial values of dv/dy, v may be made continuous, and (as has been said) a discontinuity in dv/dy does not interfere with the satisfaction of (3). If there are other places where U has the same value, dv/dy may there be either continuous or discontinuous. Even when there is but one place where $n+kU=0$ with the proposed value of n, it may happen that dv/dy is there continuous.

The argument above employed is not interfered with even though U is such that dU/dy is here and there discontinuous, so as to make d^2U/dy^2 infinite. At any such place the necessary condition is obtained by integrating (3) across the discontinuity. As was shown in my former paper (*loc. cit.*), it is

$$\left(\frac{n}{k}+U\right).\Delta\left(\frac{dv}{dy}\right)-\Delta\left(\frac{dU}{dy}\right).v=0, \quad\text{.................}(6)$$

Δ being the symbol of finite differences; and by (6) the corresponding sudden change in dv/dy is determined.

It appears then that any value of $-kU$ is a possible value of n. Are other real values admissible? If so, $n+kU$ is of one sign throughout. It is easy to see that if d^2U/dy^2 has throughout the same sign as $n+kU$, no solution is possible. I propose to prove that no solution is possible in any case if $n+kU$, being real, is of one sign throughout.

If U' be written for $U+n/k$, our equation (3) takes the form

$$U'\frac{d^2v}{dy^2}-v\frac{d^2U'}{dy^2}=k^2U'v, \quad\text{.........................}(7)$$

or on integration with respect to y,

$$U'\frac{dv}{dy}-v\frac{dU'}{dy}=K+k^2\int_0^y U'v\,dy, \quad\text{.................}(8)$$

where K is an arbitrary constant. Assume $v=U'v'$; then

$$\frac{dv'}{dy}=\frac{K}{U'^2}+\frac{k^2}{U'^2}\int_0^y v'U'^2\,dy; \quad\text{......................}(9)$$

* Graphically, the equation directs us with what curvature to proceed at any point already reached.

whence, on integration and replacement of v,

$$v = HU' + KU' \int_0^y \frac{dy}{U'^2} + k^2 U' \int_0^y \frac{dy}{U'^2} \int_0^y U' v \, dy, \quad \text{.........(10)}$$

H denoting a second arbitrary constant.

In (10) we may suppose y measured from the first wall, where $v = 0$. Hence, unless U' vanish with y, $H = 0$. Also from (8) when $y = 0$,

$$\left(U' \frac{dv}{dy}\right)_0 = K. \quad \text{............................(11)}$$

Let us now trace the course of v as a function of y, starting from the wall where $y = 0$, $v = 0$; and let us suppose first that U' is everywhere positive. By (11) K has the same sign as $(dv/dy)_0$, that is the same sign as the early values of v. Whether this sign be positive or negative, v as determined by (10) cannot again come to zero. If, for example, the initial values of v are positive, both (remaining) terms in (10) necessarily continue positive; while if v begins by being negative, it must remain finitely negative. Similarly, if U' be everywhere negative, so that K has the opposite sign to that of the early values of v, it follows that v cannot again come to zero. No solution can be found unless U' somewhere vanishes, that is unless n coincides with some value of $-kU$.

In the above argument U', and therefore also n, is supposed to be *real*, but the formula (10) itself applies whether n be real or complex. It is of special value when k is very small, that is when the wave-length along x of the disturbance is very great; for it then gives v explicitly in the form

$$v = K(U + n/k) \int_0^y \frac{dy}{(U + n/k)^2}. \quad \text{.................(12)}$$

When k is small, but not so small as to justify (12), a second approximation might be found by substituting from (12) in the last term of (10).

If we suppose in (12) that the second wall is situated at $y = l$, n is determined by

$$\int_0^l \frac{dy}{(U + n/k)^2} = 0. \quad \text{.........................(13)}$$

The integrals (12), (13) must not be taken through a place where $U + n/k = 0$, as appears from (8). We have already seen that any value of n for which this can occur is admissible. But (13) shows that no other real value of n is admissible; and it serves to determine any complex values of n.

In (13) suppose (as before) that $n/k = p + iq$; then separating the real and imaginary parts, we get

$$\int_0^l \frac{(p + U)^2 - q^2}{\{(p + U)^2 + q^2\}^2} dy = 0, \quad \int_0^l \frac{q(p + U)\, dy}{\{(p + U)^2 + q^2\}^2} = 0, \quad \text{........(14)}$$

from the second of which we may infer that if q be finite, $p+U$ must change sign, as we have already seen that it must do when $q=0$. In every case then, when k is small, the *real part* of n must equal some value of $-kU$*.

It may be of interest to show the application of (13) to a case formerly treated† in which the velocity-curve is made up of straight portions and is anti-symmetrical with respect to the point lying midway between the two walls, now taken as origin of y. Thus on the positive side

from $\quad y=0 \quad$ to $y=\frac{1}{2}b'$, $\quad U=\dfrac{Vy}{\frac{1}{2}b'}$;

from $\quad y=\frac{1}{2}b'$ to $y=\frac{1}{2}b'+b$, $\quad U=\dfrac{Vy}{\frac{1}{2}b'}+\mu V(y-\frac{1}{2}b')$;

while on the negative side U takes symmetrically the opposite values. Then if we write $n/kV=n'$, (13) becomes

$$0=\int_0^{\frac{1}{2}b'}\frac{dy}{(2y/b+n')^2}+\int_{\frac{1}{2}b'}^{\frac{1}{2}b'+b}\frac{dy}{\{2y/b+\mu(y-\frac{1}{2}b')+n'\}^2}$$

$$+ \text{ same with } n' \text{ reversed.}$$

Effecting the integrations, we find after reduction

$$n'^2=\frac{n^2}{k^2V^2}=\frac{2b+b'+2\mu b(b+b')+\mu^2b^2b'}{2b+b'}, \quad \text{............(15)}$$

in agreement with equation (23) of the paper referred to when k is there made small. Hence n, if imaginary at all, is a pure imaginary, and it is imaginary only when μ lies between $-1/b$ and $-1/b-2/b'$. The regular motion is then exponentially unstable.

In the only unstable cases hitherto investigated the velocity-curve is made up of straight portions meeting at finite angles, and it may perhaps be thought that the instability has its origin in this discontinuity. The method now under discussion disposes of any doubt. For obviously in (13) it can make no important difference whether dU/dy is discontinuous or not. If a motion is definitely unstable in the former case, it cannot become stable merely by easing off the finite angles in the velocity-curve. There exist, therefore, exponentially unstable motions in which both U and dU/dy are continuous. And it is further evident that any proposed velocity-curve may be replaced approximately by straight lines as in my former papers.

* By the method of a former paper "On the question of the Stability of the Flow of Fluids" (*Phil. Mag.* Vol. XXXIV. p. 59 (1892); *Scientific Papers*, Vol. III. p. 579) the conclusion that $p+U$ must change sign may be extended to the problem of the simple shearing motion between two parallel walls of a *viscous* fluid, and this whatever may be the value of k.

† *Proc. Lond. Math. Soc.* Vol. XIX. p. 67 (1887); *Scientific Papers*, Vol. III. p. 20, figs. (3), (4), (5).

The fact that n in equation (15) appears only as n^2 is a simple consequence of the anti-symmetrical character of U. For if in (13) we measure y from the centre and integrate between the limits $\pm\frac{1}{2}l$, we obtain in that case

$$\int_0^{\frac{1}{2}l} \frac{n^2/k^2 + U^2}{(n^2/k^2 - U^2)^2}\, dy = 0, \qquad \text{.........................(16)}$$

in which only n^2 occurs. But it does not appear that n^2 is necessarily real, as happens in (15).

Apart from such examples as were treated in my former papers in which d^2U/dy^2 vanishes except at certain definite places, there are very few cases in which (3) can be solved analytically. If we suppose that $v = \sin(\pi y/l)$, vanishing when $y = 0$ and when $y = l$, and seek what is then admissible for U, we get

$$U + n/k = A \cos\{k^2 + \pi^2/l^2\}^{\frac{1}{2}} y + B \sin\{k^2 + \pi^2/l^2\}^{\frac{1}{2}} y, \qquad \text{......(17)}$$

in which A and B are arbitrary and n may as well be supposed to be zero. But since U varies with k, the solution is of no great interest.

In estimating the significance of our results respecting stability, we must of course remember that the disturbance has been assumed to be and to remain infinitely small. Where stability is indicated, the magnitude of the admissible disturbance may be very restricted. It was on these lines that Kelvin proposed to explain the apparent contradiction between theoretical results for an inviscid fluid and observation of what happens in the motion of real fluids which are all more or less viscous. Prof. McF. Orr has carried this explanation further *. Taking the case of a simple shearing motion between two walls, he investigates a composite disturbance, periodic with respect to x but not with respect to t, given initially as

$$v = B \cos lx \cos my, \qquad \text{..........................(18)}$$

and he finds, equation (38), that when m is large the disturbance may increase very much, though ultimately it comes to zero. Stability in the mathematical sense (B infinitely small) may thus be not inconsistent with a practical instability. A complete theoretical proof of instability requires not only a method capable of dealing with finite disturbances but also a definition, not easily given, of what is meant by the term. In the case of stability we are rather better situated, since by absolute stability we may understand complete recovery from disturbances of any kind however large, such as Reynolds showed to occur in the present case when viscosity is paramount †. In the absence of dissipation, stability in this sense is not to be expected.

* *Proc. Roy. Irish Academy*, Vol. XXVII. Section A, No. 2, 1907. Other related questions are also treated.

† See also Orr, *Proc. Roy. Irish Academy*, 1907, p. 124.

Another manner of regarding the present problem of the shearing motion of an inviscid fluid is instructive. In the original motion the vorticity is constant throughout the whole space between the walls. The disturbance is represented by a superposed vorticity, which may be either positive or negative, and this vorticity everywhere *moves with the fluid*. At any subsequent time the same vorticities exist as initially; the only question is as to their distribution. And when this distribution is known, the whole motion is determined. Now it would seem that the added vorticities will produce most effect if the positive parts are brought together, and also the negative parts, as much as is consistent with the prescribed periodicity along x, and that even if this can be done the effect cannot be out of proportion to the magnitude of the additional vorticities. If this view be accepted, the temporary large increase in Prof. Orr's example would be attributed to a specially unfavourable distribution initially in which *on larger* the positive and negative parts of the added vorticities are closely intermingled. We may even go further and regard the subsequent tendency to evanescence, rather than the temporary increase, as the normal phenomenon. The difficulty in reconciling the observed behaviour of actual fluids with the theory of an inviscid fluid still seems to me to be considerable, unless indeed we can admit a distinction between a fluid of infinitely small viscosity and one of none at all.

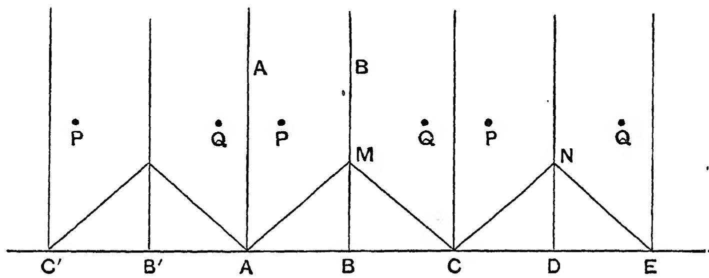
At one time I thought that the instability suggested by observation might attach to the stages through which a viscous liquid must pass in order to acquire a uniform shearing motion rather than to the final state itself. Thus in order to find an explanation of "skin friction" we may suppose the fluid to be initially at rest between two infinite fixed walls, one of which is then suddenly made to move in its own plane with a uniform velocity. In the earlier stages the other wall has no effect and the problem is one considered by Fourier in connexion with the conduction of heat. The velocity U in the laminar motion satisfies generally an equation of the form

$$\frac{dU}{dt} = \nu \frac{d^2U}{dy^2}, \qquad \text{.............................}(19)$$

with the conditions that initially $(t = 0)$ $U = 0$, and that from $t = 0$ onwards $U = 1$ when $y = 0$, and (if we please) $U = 0$ when $y = l$. We might employ Fourier's solution, but all that we require follows at once from the differential equation itself. It is evident that dU/dt, and therefore d^2U/dy^2, is everywhere positive and accordingly that a non-viscous liquid, moving laminarly as the viscous fluid moves in any of these stages, is stable. It would appear then that no explanation is to be found in this direction.

Hitherto we have supposed that the disturbance is periodic as regards x, but a simple example, not coming under this head, may be worthy of notice. It is that of the disturbance due to a single vortex filament in which the

vorticity differs from the otherwise uniform vorticity of the neighbouring fluid. In the figure the lines *AA*, *BB* represent the situation of the walls and *AM* the velocity-curve of the original shearing motion rising from zero at *A* to a finite value at *M*. For the present purpose, however, we suppose material walls to be absent, but that the same effect (of prohibiting normal motion) is arrived at by suitable suppositions as to the fluid lying outside and now imagined infinite. It is only necessary to continue the velocity-curve in the manner shown *AMCN*..., the vorticities in the alternate layers of equal width being equal and opposite. Symmetry then shows that under the operation of these vorticities the fluid moves as if *AA*, *BB*, &c. were material walls.



We have now to trace the effect of an additional vorticity, supposed positive, at a point *P*. If the wall *AA* were alone concerned, its effect would be imitated by the introduction of an opposite vorticity at the point *Q* which is the image of *P* in *AA*. Thus *P* would move under the influence of the original vorticities, already allowed for, and of the negative vorticity at *Q*. Under the latter influence it would move parallel to *AA* with a certain velocity, and for the same reason *Q* would move similarly, so that *PQ* would remain perpendicular to *AA*. To take account of both walls the more complicated arrangement shown in the figure is necessary, in which the points *P* represent equal positive vorticities and *Q* equal negative vorticities. The conditions at both walls are thus satisfied; and as before all the vortices *P*, *Q* move under each other's influence so as to remain upon a line perpendicular to *AA*. Thus, to go back to the original form of the problem, *P* moves parallel to the walls with a constant velocity, and no change ensues in the character of the motion—a conclusion which will appear the more remarkable when we remember that there is no limitation upon the magnitude of the added vorticity.

The same method is applicable—in imagination at any rate—whatever be the distribution of vorticities between the walls, and the corresponding velocity at any point is determined by quadratures on Helmholtz's principle. The new positions of all the vorticities after a short time are thus found, and then a new departure may be taken, and so on indefinitely.

378.

REFLECTION OF LIGHT AT THE CONFINES OF A DIFFUSING MEDIUM.

[*Nature*, Vol. XCII. p. 450, 1913.]

I SUPPOSE that everyone is familiar with the beautifully graded illumination of a paraffin candle, extending downwards from the flame to a distance of several inches. The thing is seen at its best when there is but one candle in an otherwise dark room, and when the eye is protected from the direct light of the flame. And it must often be noticed when a candle is broken across, so that the two portions are held together merely by the wick, that the part below the fracture is much darker than it would otherwise be, and the part above brighter, the contrast between the two being very marked. This effect is naturally attributed to reflection, but it does not at first appear that the cause is adequate, seeing that at perpendicular incidence the reflection at the common surface of wax and air is only about 4 per cent.

A little consideration shows that the efficacy of the reflection depends upon the incidence not being limited to the neighbourhood of the perpendicular. In consequence of diffusion* the propagation of light within the wax is not specially along the length of the candle, but somewhat approximately equal in *all* directions. Accordingly at a fracture there is a good deal of "total reflection." The general attenuation downwards is doubtless partly due to defect of transparency, but also, and perhaps more, to the lateral escape of light at the surface of the candle, thereby rendered visible. By hindering this escape the brightly illuminated length may be much increased.

The experiment may be tried by enclosing the candle in a reflecting tubular envelope. I used a square tube composed of four rectangular pieces of mirror glass, 1 in. wide, and 4 or 5 in. long, held together by strips of

* To what is the diffusion due? Actual cavities seem improbable. Is it chemical heterogeneity, or merely varying orientation of chemically homogeneous material operative in virtue of double refraction?

pasted paper. The tube should be lowered over the candle until the whole of the flame projects, when it will be apparent that the illumination of the candle extends decidedly lower down than before.

In imagination we may get quit of the lateral loss by supposing the diameter of the candle to be increased without limit, the source of light being at the same time extended over the whole of the horizontal plane.

To come to a definite question, we may ask what is the proportion of light reflected when it is incident equally in all directions upon a surface of transition, such as is constituted by the candle fracture. The answer depends upon a suitable integration of Fresnel's expression for the reflection of light of the two polarisations, viz.

$$S^2 = \frac{\sin^2(\theta-\theta')}{\sin^2(\theta+\theta')}, \qquad T^2 = \frac{\tan^2(\theta-\theta')}{\tan^2(\theta+\theta')}, \quad \text{..............(1)}$$

where θ, θ' are the angles of incidence and refraction. We may take first the case where $\theta > \theta'$, that is, when the transition is from the less to the more refractive medium.

The element of solid angle is $2\pi \sin\theta \, d\theta$, and the area of cross-section corresponding to unit area of the refracting surface is $\cos\theta$; so that we have to consider

$$2\int_0^{\frac{1}{2}\pi} \sin\theta\cos\theta\,(S^2 \text{ or } T^2)\,d\theta, \quad \text{....................(2)}$$

the multiplier being so chosen as to make the integral equal to unity when S^2 or T^2 has that value throughout. The integral could be evaluated analytically, at any rate in the case of S^2, but the result would scarcely repay the trouble. An estimate by quadratures in a particular case will suffice for our purposes, and to this we shall presently return.

In (2) θ varies from 0 to $\frac{1}{2}\pi$ and θ' is always real. If we suppose the passage to be in the other direction, viz. from the more to the less refractive medium, S^2 and T^2, being symmetrical in θ and θ', remain as before, and we have to integrate

$$2\sin\theta'\cos\theta'\,(S^2 \text{ or } T^2)\,d\theta'.$$

The integral divides itself into two parts, the first from 0 to α, where α is the critical angle corresponding to $\theta = \frac{1}{2}\pi$. In this S^2, T^2 have the values given in (1). The second part of the range from $\theta' = \alpha$ to $\theta' = \frac{1}{2}\pi$ involves "total reflection," so that S^2 and T^2 must be taken equal to unity. Thus altogether we have

$$2\int_0^{\alpha} \sin\theta'\cos\theta'\,(S^2 \text{ or } T^2)\,d\theta' + 2\int_{\alpha}^{\frac{1}{2}\pi} \sin\theta'\cos\theta'\,d\theta', \quad \text{......(3)}$$

in which $\sin \alpha = 1/\mu$, μ (greater than unity) being the refractive index. In (3)

$$2 \sin \theta' \cos \theta' \, d\theta' = d \sin^2 \theta' = \mu^{-2} d \sin^2 \theta,$$

and thus

$$(3) = \mu^{-2} \times (2) + 1 - \mu^{-2} = \frac{1}{\mu^2} \left\{ \mu^2 - 1 + \int_0^{\frac{1}{2}\pi} \sin 2\theta \, (S^2 \text{ or } T^2) \, d\theta \right\}, \ldots (4)$$

expressing the proportion of the uniformly diffused incident light reflected in this case.

Much the more important part is the light totally reflected. If $\mu = 1{\cdot}5$, this amounts to $5/9$ or $0{\cdot}5556$.

With the same value of μ, I find by Weddle's rule

$$\int_0^{\frac{1}{2}\pi} \sin 2\theta \, . \, S^2 d\theta = 0{\cdot}1460, \quad \int_0^{\frac{1}{2}\pi} \sin 2\theta \, . \, T^2 d\theta = 0{\cdot}0339.$$

Thus for light vibrating perpendicularly to the plane of incidence

$$(4) = 0{\cdot}5556 + 0{\cdot}0649 = 0{\cdot}6205\,;$$

while for light vibrating in the plane of incidence

$$(4) = 0{\cdot}5556 + 0{\cdot}0151 = 0{\cdot}5707.$$

The increased reflection due to the diffusion of the light is thus abundantly explained, by far the greater part being due to the total reflection which ensues when the incidence in the denser medium is somewhat oblique.

THE PRESSURE OF RADIATION AND CARNOT'S PRINCIPLE.

[*Nature*, Vol. XCII. pp. 527, 528, 1914.]

As is well known, the pressure of radiation, predicted by Maxwell, and since experimentally confirmed by Lebedew and by Nichols and Hull, plays an important part in the theory of radiation developed by Boltzmann and W. Wien. The existence of the pressure according to electromagnetic theory is easily demonstrated*, but it does not appear to be generally remembered that it could have been deduced with some confidence from thermodynamical principles, even earlier than in the time of Maxwell. Such a deduction was, in fact, made by Bartoli in 1876, and constituted the foundation of Boltzmann's work†. Bartoli's method is quite sufficient for his purpose; but, mainly because it employs irreversible operations, it does not lend itself to further developments. It may therefore be of service to detail the elementary argument on the lines of Carnot, by which it appears that in the absence of a pressure of radiation it would be possible to raise heat from a lower to a higher temperature.

The imaginary apparatus is, as in Boltzmann's theory, a cylinder and piston formed of perfectly reflecting material, within which we may suppose the radiation to be confined. This radiation is always of the kind characterised as complete (or black), a requirement satisfied if we include also a very small black body with which the radiation is in equilibrium. If the operations are slow enough, the size of the black body may be reduced without limit, and then the whole energy at a given temperature is that of the radiation and proportional to the volume occupied. When we have occasion to introduce or abstract heat, the communication may be supposed

* See, for example, J. J. Thomson, *Elements of Electricity and Magnetism* (Cambridge, 1895, § 241); Rayleigh, *Phil. Mag.* Vol. XLV. p. 222 (1898); *Scientific Papers*, Vol. IV. p. 354.

† *Wied. Ann.* Vol. XXXII. pp. 31, 291 (1884). It is only through Boltzmann that I am acquainted with Bartoli's reasoning.

in the first instance to be with the black body. The operations are of two kinds: (1) compression (or rarefaction) of the kind called *adiabatic*, that is, without communication of heat. If the volume increases, the temperature must fall, even though in the absence of pressure upon the piston no work is done, since the same energy of complete radiation now occupies a larger space. Similarly a rise of temperature accompanies adiabatic contraction. In the second kind of operation (2) the expansions and contractions are *isothermal*—that is, without change of temperature. In this case heat must pass into the black body when the volume expands and out of it when the volume contracts, and at a given temperature the amount of heat which must pass is proportional to the change of volume.

The cycle of operations to be considered is the same as in Carnot's theory, the only difference being that here, in the absence of pressure, there is no question of external work. Begin by isothermal expansion at the lower temperature during which heat is taken in. Then compress adiabatically until a higher temperature is reached. Next continue the compression isothermally until the same amount of heat is given out as was taken in during the first expansion. Lastly, restore the original volume adiabatically. Since no heat has passed upon the whole in either direction, the final state is identical with the initial state, the temperature being recovered as well as the volume. The sole result of the cycle is that heat is raised from a lower to a higher temperature. Since this is assumed to be impossible, the supposition that the operations can be performed without external work is to be rejected—in other words, we must regard the radiation as exercising a pressure upon the moving piston. Carnot's principle and the absence of a pressure are incompatible.

For a further discussion it is, of course, desirable to employ the general formulation of Carnot's principle, as in a former paper*. If p be the pressure, θ the absolute temperature,

$$\theta \frac{dp}{d\theta} = M, \quad \text{.................................(29)}$$

where $M\,dv$ represents the heat that must be communicated, while the volume alters by dv and $d\theta = 0$. In the application to radiation M cannot vanish, and therefore p cannot. In this case clearly

$$M = U + p. \quad \text{.................................(30)}$$

where U denotes the volume-density of the energy—a function of θ only. Hence

$$\theta \frac{dp}{d\theta} = U + p. \quad \text{.................................(31)}$$

* "On the Pressure of Vibrations," *Phil. Mag.* Vol. III. p. 338, 1902; *Scientific Papers*, Vol. v. p. 47.

If we assume from electromagnetic theory that

$$p = \tfrac{1}{3} U, \quad \text{.................................. (32)}$$

it follows at once that

$$U \propto \theta^4, \quad \text{.................................. (33)}$$

the well-known law of Stefan.

In (31) if p be known as a function of θ, U as a function of θ follows immediately. If, on the other hand, U be known, we have

$$d\left(\frac{p}{\theta}\right) = \frac{U}{\theta^2}\, d\theta,$$

and thence

$$\frac{p}{\theta} = \int_0^\theta \frac{U}{\theta^2}\, d\theta + C. \quad \text{.................................. (34)}$$

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FURTHER APPLICATIONS OF BESSEL'S FUNCTIONS OF HIGH ORDER TO THE WHISPERING GALLERY AND ALLIED PROBLEMS.

[*Philosophical Magazine*, Vol. XXVII. pp. 100—109, 1914.]

IN the problem of the Whispering Gallery* waves in two dimensions, of length small in comparison with the circumference, were shown to run round the concave side of a wall with but little tendency to spread themselves inwards. The wall was supposed to be perfectly reflecting for all kinds of waves. But the question presents itself whether the sensibly perfect reflexion postulated may not be attained on the principle of so-called "total reflexion," the wall being merely the transition between two uniform media of which the outer is the less refracting. It is not to be expected that absolutely no energy should penetrate and ultimately escape to an infinite distance. The analogy is rather with the problem treated by Stokes† of the communication of vibrations from a vibrating solid, such as a bell or wire, to a surrounding gas, when the wave-length in the gas is somewhat large compared with the dimensions of the vibrating segments. The energy radiated to a distance may then be extremely small, though not mathematically evanescent.

A comparison with the simple case where the surface of the vibrating body is *plane* ($x = 0$) is interesting, especially as showing how the partial

* *Phil. Mag.* Vol. xx. p. 1001 (1910); *Scientific Papers*, Vol. v. p. 619. But the numbers there given require some correction owing to a slip in Nicholson's paper from which they were derived, as was first pointed out to me by Prof. Macdonald. Nicholson's table should be interpreted as relating to the values, not of $2{\cdot}1123\,(n-z)/z^{\frac{1}{3}}$, but of $1{\cdot}3447\,(n-z)/z^{\frac{1}{3}}$, see Nicholson, *Phil. Mag.* Vol. xxv. p. 200 (1913). Accordingly, in my equation (5) $1{\cdot}1814\,n^{\frac{1}{3}}$ should read $1{\cdot}8558\,n^{\frac{1}{3}}$, and in equation (8) $\cdot 51342\,n^{\frac{1}{3}}$ should read $\cdot 8065\,n^{\frac{1}{3}}$. [1916. Another error should be noticed. In (3), $\int_0^\pi \cos n(\omega - \sin\omega)\,d\omega/\pi$ must be omitted, the integrand being periodic. See Watson, *Phil. Mag.* Vol. xxxii. p. 233, 1916.]

† *Phil. Trans.* 1868. See *Theory of Sound*, Vol. II. § 324.

escape of energy is connected with the curvature of the surface. If V be the velocity of propagation, and $2\pi/k$ the wave-length of plane waves of the given period, the time-factor is e^{ikVt}, and the equation for the velocity-potential in two dimensions is

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + k^2\phi = 0. \quad \text{.........................(1)}$$

If ϕ be also proportional to $\cos my$, (1) reduces to

$$\frac{d^2\phi}{dx^2} + (k^2 - m^2)\,\phi = 0, \quad \text{.........................(2)}$$

of which the solution changes its form when m passes through the value k. For our purpose m is to be supposed greater than k, viz. the wave-length of plane waves is to be greater than the linear period along y. That solution of (1) on the positive side which does not become infinite with x is proportional to $e^{-x\sqrt{(m^2-k^2)}}$, so that we may take

$$\phi = \cos kVt \,.\, \cos my \,.\, e^{-x\sqrt{(m^2-k^2)}}. \quad \text{....................(3)}$$

However the vibration may be generated at $x = 0$, provided only that the linear period along y be that assigned, it is limited to relatively small values of x and, since no energy can escape, no work is done on the whole at $x = 0$. And this is true by however little m may exceed k.

The reason of the difference which ensues when the vibrating surface is curved is now easily seen. Suppose, for example, that in two dimensions ϕ is proportional to $\cos n\theta$, where θ is a vectorial angle. Near the surface of a cylindrical vibrator the conditions may be such that (3) is approximately applicable, and ϕ rapidly diminishes as we go outwards. But when we reach a radius vector r which is sensibly different from the initial one, the conditions may change. In effect the linear dimension of the vibrating compartment increases proportionally to r, and ultimately the equation (2) changes its form and ϕ oscillates, instead of continuing an exponential decrease. *Some* energy always escapes, but the amount must be very small if there is a sufficient margin to begin with between m and k.

It may be well before proceeding further to follow a little more closely what happens when there is a transition at a plane surface $x = 0$ from a more to a less refractive medium. The problem is that of total reflexion when the incidence is grazing, in which case the usual formulæ* become nugatory. It will be convenient to fix ideas upon the case of sonorous waves, but the results are of wider application. The general differential equation is of the form

$$\frac{d^2\phi}{dt^2} = V^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right), \quad \text{.........................(4)}$$

* See for example *Theory of Sound*, Vol. II. § 270.

which we will suppose to be adapted to the region where x is negative. On the right (x positive) V is to be replaced by V_1, where $V_1 > V$, and ϕ by ϕ_1. In optical notation $V_1/V = \mu$, where μ (greater than unity) is the refractive index. We suppose ϕ and ϕ_1 to be proportional to $e^{i(by+ct)}$, b and c being the same in both media. Further, on the left we suppose b and c to be related as they would be for simple plane waves propagated parallel to y. Thus (4) becomes, with omission of $e^{i(by+ct)}$,

$$\frac{d^2\phi}{dx^2} = 0, \quad \frac{d^2\phi_1}{dx^2} = b^2(\mu^2 - 1)/\mu^2, \quad \text{.....................(5)}$$

of which the solutions are

$$\phi = A + Bx, \qquad \phi_1 = Ce^{-bx\sqrt{(\mu^2-1)}/\mu}, \quad \text{..................(6)}$$

A, B, C denoting constants so far arbitrary. The boundary conditions require that when $x = 0$, $d\phi/dx = d\phi_1/dx$ and that $\rho\phi = \rho_1\phi_1$, ρ, ρ_1 being the densities. Hence discarding the imaginary part, and taking $A = 1$, we get finally

$$\phi = \left\{1 - \frac{\rho bx\sqrt{(\mu^2-1)}}{\rho_1\mu}\right\}\cos(by + ct), \quad \text{..................(7)}$$

$$\phi_1 = \frac{\rho}{\rho_1} e^{-bx\sqrt{(\mu^2-1)}/\mu}\cos(by + ct). \quad \text{........................(8)}$$

It appears that while nothing can escape on the positive side, the amplitude on the negative side increases rapidly as we pass away from the surface of transition.

If $\mu < 1$, a wave of the ordinary kind is propagated into the second medium, and energy is conveyed away.

In proceeding to consider the effect of curvature it will be convenient to begin with Stokes' problem, taking advantage of formulæ relating to Bessel's and allied functions of high order developed by Lorenz, Nicholson, and Macdonald*. The motion is supposed to take place in two dimensions, and ideas may be fixed upon the case of aerial vibrations. The velocity-potential ϕ is expressed by means of polar coordinates r, θ, and will be assumed to be proportional to $\cos n\theta$, attention being concentrated upon the case where n is a large integer. The problem is to determine the motion at a distance due to the normal vibration of a cylindrical surface at $r = a$, and it turns upon the character of the function of r which represents a disturbance propagated outwards. If $D_n(kr)$ denote this function, we have

$$\phi = e^{ikVt}\cos n\theta \, . \, D_n(kr), \quad \text{........................(9)}$$

and $D_n(z)$ satisfies Bessel's equation

$$D_n'' + \frac{1}{z}D_n' + \left(1 - \frac{n^2}{z^2}\right)D_n = 0. \quad \text{..................(10)}$$

* Compare also Debye, *Math. Ann.* Vol. LXVII. (1909).

It may be expressed in the form

$$D_n = \frac{J_{-n} - e^{in\pi} J_n}{\sin n\pi}, \quad \text{.........................(11)}$$

which, however, requires a special evaluation when n is an integer. Using Schläfli's formula

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - n\theta)\,d\theta - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-n\theta - z\sinh\theta}\,d\theta, \quad \text{...(12)}$$

n being positive or negative, and z positive, we find

$$D_n(z) = \frac{1}{\pi}\int_0^\infty e^{n\theta - z\sinh\theta}\,d\theta + \frac{\cos n\pi}{\pi}\int_0^\infty e^{-n\theta - z\sinh\theta}\,d\theta$$

$$- \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - n\theta)\,d\theta - \frac{i}{\pi}\int_0^\pi \cos(z\sin\theta - n\theta)\,d\theta, \quad \text{......(13)}$$

the imaginary part being $-iJ_n(z)$ simply. This holds good for any integral value of n. The present problem requires the examination of the form assumed by D_n when n is very great and the ratio z/n decidedly greater, or decidedly less, than unity.

In the former case we set $n = z\sin\alpha$, and the important part of D_n arises from the two integrals last written. It appears* that

$$D_n = \left(\frac{2}{\pi z\cos\alpha}\right)^{\frac{1}{2}} e^{-i\rho}, \quad \text{.......................(14)}$$

where $$\rho = \tfrac{1}{4}\pi + z\{\cos\alpha - (\tfrac{1}{2}\pi - \alpha)\sin\alpha\}, \quad \text{................(15)}$$

or when z is extremely large ($\alpha = 0$)

$$D_n(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(\frac{1}{4}\pi + z)}. \quad \text{.......................(16)}$$

At a great distance the value of ϕ in (9) thus reduces to

$$\phi = \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \cos n\theta \,.\, e^{i\{k(Vt-r) - \frac{1}{4}\pi\}}, \quad \text{.................(17)}$$

from which finally the imaginary part may be omitted.

When on the other hand z/n is decidedly less than unity, the most important part of (13) arises from the first and last integrals. We set $n = z\cosh\beta$, and then, n being very great,

$$D_n(z) = \left(\frac{\coth\beta}{2n\pi}\right)^{\frac{1}{2}} \{2e^{-t} - ie^{t}\}, \quad \text{.................(18)}$$

where $$t = n(\tanh\beta - \beta). \quad \text{.........................(19)}$$

* Nicholson, *B. A. Report*, Dublin, 1908, p. 595; *Phil. Mag.* Vol. XIX. p. 240 (1910); Macdonald, *Phil. Trans.* Vol. CCX. p. 135 (1909).

Also, the most important part of the real and imaginary terms being retained,

$$D_n'(z) = -\left(\frac{\sinh\beta\cosh\beta}{2n\pi}\right)^{\frac{1}{2}} \{2e^{-t} + ie^{t}\}. \quad\text{.............(20)}$$

The application is now simple. From (9) with introduction of an arbitrary coefficient

$$\frac{d\phi}{dr} = kA\, e^{ikVt} \cos n\theta \,.\, D_n'(kr). \quad\text{....................(21)}$$

If we suppose that the normal velocity of the vibrating cylindrical surface $(r = a)$ is represented by $e^{ikVt}\cos n\theta$, we have

$$kAD_n'(ka) = 1, \quad\text{............................(22)}$$

and thus at distance r

$$\phi = e^{ikVt} \cos n\theta \,\frac{D_n(kr)}{kD_n'(ka)}, \quad\text{....................(23)}$$

or when r is very great

$$\phi = \cos n\theta \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \frac{e^{i\{k(Vt-r)-\frac{1}{4}\pi\}}}{kD_n'(ka)}. \quad\text{..................(24)}$$

We may now, following Stokes, compare the actual motion at a distance with that which would ensue were lateral motion prevented, as by the insertion of a large number of thin plane walls radiating outwards along the lines $\theta =$ constant, the normal velocity at $r = a$ being the same in both cases. In the altered problem we have merely in (23) to replace D_n, D_n' by D_0, D_0'. When z is great enough, $D_n(z)$ has the value given in (16), independently of the particular value of n. Accordingly the ratio of velocity-potentials at a distance in the two cases is represented by the symbolic fraction

$$\frac{D_0'(ka)}{D_n'(ka)}, \quad\text{..............................(25)}$$

in which

$$D_0'(ka) = -i\left(\frac{2}{\pi ka}\right)^{\frac{1}{2}} e^{-i(\frac{1}{4}\pi + ka)}. \quad\text{.................(26)}$$

We have now to introduce the value of $D_n'(ka)$. When n is very great, and ka/n decidedly less than unity, t is negative in (20), and e^t is negligible in comparison with e^{-t}. The modulus of (25) is therefore

$$\left(\frac{n/ka}{\sinh\beta\cosh\beta}\right)^{\frac{1}{2}} e^{t}, \quad\text{or}\quad \frac{e^{-n(\beta-\tanh\beta)}}{\sinh^{\frac{1}{2}}\beta}. \quad\text{..............(27)}$$

For example, if $n = 2ka$, so that the linear period along the circumference of the vibrating cylinder $(2\pi a/n)$ is half the wave-length,

$$\cosh\beta = 2, \quad \beta = 1{\cdot}317, \quad \sinh\beta = 1{\cdot}7321, \quad \tanh\beta = {\cdot}8660,$$

and the numerical value of (27) is

$$e^{-{\cdot}4510n} \div \sqrt{(1{\cdot}732)}.$$

When n is great, the vibration at a distance is extraordinarily small in comparison with what it would have been were lateral motion prevented. As another example, let $n = 1{\cdot}1\,ka$. Then (27) $= e^{-{\cdot}027n} \div \sqrt{({\cdot}4587)}$. Here n would need to be about 17 times larger for the same sort of effect.

The extension of Stokes' analysis to large values of n only emphasizes his conclusion as to the insignificance of the effect propagated to a distance when the vibrating segments are decidedly smaller than the wave-length.

We now proceed to the case of the whispering gallery supposed to act by "total reflexion." From the results already given, we may infer that when the refractive index is moderate, the escape of energy must be very small, and accordingly that the vibrations inside have long persistence. There is, however, something to be said upon the other side. On account of the concentration near the reflecting wall, the store of energy to be drawn upon is diminished. At all events the problem is worthy of a more detailed examination.

Outside the surface of transition ($r = a$) we have the same expression (9) as before for the velocity-potential, k and V having values proper to the outer medium. Inside k and V are different, but the product kV is the same. We will denote the altered k by h. In accordance with our suppositions $h > k$, and h/k represents the refractive index (μ) of the inside medium relatively to that outside. On account of the damping k and h are complex, though their ratio is real; but the imaginary part is relatively small. Thus, omitting the factors $e^{ikVt} \cos n\theta$, we have ($r > a$)

$$\phi = A D_n(kr), \qquad \text{.............................(28)}$$

and inside ($r < a$)

$$\phi = B J_n(hr). \qquad \text{.............................(29)}$$

The boundary conditions to be satisfied when $r = a$ are easily expressed. The equality of normal motions requires that

$$kAD_n'(ka) = hB J_n'(ha); \qquad \text{.......................(30)}$$

and the equality of pressures requires that

$$\sigma A D_n(ka) = \rho B J_n(ha), \qquad \text{.......................(31)}$$

σ, ρ being the densities of the outer and inner media respectively. The equation for determining the values of ha, ka (in addition to $h/k = \mu$) is accordingly

$$\frac{kD_n'(ka)}{\sigma D_n(ka)} = \frac{hJ_n'(ha)}{\rho J_n(ha)}. \qquad \text{.......................(32)}$$

Equation (32) cannot be satisfied exactly by real values of h and k; for, although J_n'/J_n is then real, D_n'/D_n includes an imaginary part. But since the imaginary part is relatively small, we may conclude that *approximately* h and k are real, and the first step is to determine these real values.

Since ka is supposed to be decidedly less than n, D_n and D_n' are given by (18), (20), and, if we neglect the imaginary part,

$$\frac{D_n'(ka)}{D_n(ka)} = -\sinh\beta. \quad \text{..........................(33)}$$

Thus (32) becomes
$$\frac{J_n'(ha)}{J_n(ha)} = -\frac{\rho k}{\sigma h}\sinh\beta, \quad \text{........................(34)}$$

the right-hand member being real and negative. Of this a solution can always be found in which $ha = n$ very nearly. For* $J_n(z)$ increases with z from zero until $z = n + \cdot 8065 n^{\frac{1}{3}}$, when $J_n'(z) = 0$, and then decreases until it vanishes when $z = n + 1\cdot 8558 n^{\frac{1}{3}}$. Between these limits for z, J_n'/J_n assumes all possible negative values. Substituting n for ha on the right in (34), we get

$$-\frac{\rho k a}{\sigma n}\sinh\beta, \quad \text{or} \quad -\frac{\rho}{\sigma}\tanh\beta, \quad \text{..................(35)}$$

while $\cosh\beta = \mu$. The approximate real value of ha is thus n simply, while that of ka is n/μ.

These results, though stated for aerial vibrations, have as in all such (two-dimensional) cases a wider application, for example to electrical vibrations, whether the electric force be in or perpendicular to the plane of r, θ. For ordinary gases, of which the compressibility is the same,

$$\rho/\sigma = h^2/k^2 = \mu^2.$$

Hitherto we have neglected the small imaginary part of D_n'/D_n. By (18), (20), when z is real,

$$\frac{D_n'(z)}{D_n(z)} = -\sinh\beta\,\frac{2e^{-\gamma} + ie^{\gamma}}{2e^{-\gamma} - ie^{\gamma}} = -\sinh\beta\,.\,(1 + ie^{2\gamma}) \quad \text{........(36)}$$

approximately, with $\cosh\beta = n/z$. We have now to determine what small imaginary additions must be made to ha, ka in order to satisfy the complete equation.

Let us assume $ha = x + iy$, where x and y are real, and y is small. Then approximately

$$\frac{J_n'(x+iy)}{J_n(x+iy)} = \frac{J_n'(x) + iyJ_n''(x)}{J_n(x) + iyJ_n'(x)},$$

and
$$J_n''(x) = -\frac{1}{x}J_n'(x) - \left(1 - \frac{n^2}{x^2}\right)J_n(x).$$

Since the approximate value of x is n, J_n'' is small compared with J_n or J_n', and we may take

$$\frac{J_n'(x+iy)}{J_n(x+iy)} = \frac{J_n'(x)}{J_n(x)}\left\{1 - iy\frac{J_n'(x)}{J_n(x)}\right\}. \quad \text{..............(37)}$$

* See paper quoted on p. 211 and correction.

Similarly, if we write $ka = x' + iy'$, where $x' = x/\mu$, $y' = y/\mu$,

$$\frac{D_n'(x'+iy')}{D_n(x'+iy')} = \frac{D_n'(x') + iy' D_n''(x')}{D_n(x') + iy' D_n'(x')},$$

and in virtue of (10)

$$D_n''(x') = -\frac{\cosh\beta}{n} D_n'(x') + \sinh^2\beta\, D_n(x'),$$

where $\cosh\beta = n/x'$. Thus

$$\frac{D_n'(x'+iy')}{D_n(x'+iy')} = \frac{D_n'(x')}{D_n(x')}\left\{1 + iy'\left(-\frac{\cosh\beta}{n} + \sinh^2\beta\frac{D_n}{D_n'} - \frac{D_n'}{D_n}\right)\right\}.$$

Accordingly with use of (36)

$$\frac{D_n'(x'+iy')}{D_n(x'+iy')} = -\sinh\beta\left\{1 + ie^{2t} + iy'\left(-\frac{\cosh\beta}{n} + \sinh^2\beta\frac{D_n}{D_n'} - \frac{D_n'}{D_n}\right)\right\}. \text{ (38)}$$

Equation (32) asserts the equality of the expressions on the two sides of (38) with

$$\frac{h\sigma}{kp}\frac{J_n'(x)}{J_n(x)}\left\{1 - iy\frac{J_n'(x)}{J_n(x)}\right\}. \quad \text{.....................(37)}$$

If we neglect the imaginary terms in (38), (37), we fall back on (34). The imaginary terms themselves give a second equation. In forming this we notice that the terms in y' vanish in comparison with that in y. For in the coefficient of y' the first part, viz. $-n^{-1}\cosh\beta$, vanishes when n is made infinite, while the second and third parts compensate one another in virtue of (33). Accordingly (32) gives with regard to (34)

$$y = \frac{\sigma h}{\rho k}\frac{e^{2t}}{\sinh\beta} = \frac{\mu\sigma}{\rho}\frac{e^{-2n(\beta-\tanh\beta)}}{\sinh\beta}, \quad \text{..................(39)}$$

in which

$$\cosh\beta = \mu. \quad \text{..............................(40)}$$

In (39) iy is the imaginary increment of ha, of which the principal real part is n. In the time-factor e^{ikVt}, the exponent

$$ikVt = \frac{ihaVt}{\mu a} = \frac{inVt}{\mu a}\left\{1 + \frac{i\,(39)}{n}\right\}.$$

In one complete period τ, $nVt/\mu a$ undergoes the increment 2π. The exponential factor giving the decrement in one period is thus

$$e^{-2\pi\,(39)/n}, \quad \text{..................................(41)}$$

or with regard to the smallness of (39)

$$1 - \frac{2\pi\mu\sigma}{n\rho}\frac{e^{-2n(\beta-\tanh\beta)}}{\sinh\beta}. \quad \text{........................(42)}$$

This is the factor by which the amplitude is reduced after each complete period.

In the case of ordinary gases $\rho/\sigma = \mu^2$. As an example, take $\mu = \cosh\beta = 1{\cdot}3$; then (42) gives

$$1 - {\cdot}581\, n^{-1} e^{-{\cdot}236\, n}. \qquad \text{.............................(43)}$$

When n rises beyond 10, the damping according to (43) becomes small; and when n is at all large, the vibrations have very great persistence.

In the derivation of (42) we have spoken of stationary vibrations. But the damping is, of course, the same for vibrations which progress round the circumference, since these may be regarded as compounded of two sets of stationary vibrations which differ in phase by 90°.

Calculation thus confirms the expectation that the whispering gallery effect does not require a perfectly reflecting wall, but that the main features are reproduced in transparent media, provided that the velocity of waves is moderately larger outside than inside the surface of transition. And further, the less the curvature of this surface, the smaller is the refractive index (greater than unity) which suffices.

ON THE DIFFRACTION OF LIGHT BY SPHERES OF SMALL* RELATIVE INDEX.

[*Proceedings of the Royal Society*, A, Vol. xc. pp. 219—225, 1914.]

In a short paper "On the Diffraction of Light by Particles Comparable with the Wave-length†," Keen and Porter describe curious observations upon the intensity and colour of the light transmitted through small particles of precipitated sulphur, while still in a state of suspension, when the size of the particles is comparable with, or decidedly larger than, the wave-length of the light. The particles principally concerned in their experiments appear to have decidedly exceeded those dealt with in a recent paper‡, where the calculations were pushed only to the point where the circumference of the sphere is 2·25 λ. The authors cited give as the size of the particles, when the intensity of the light passing through was a minimum, 6 μ to 10 μ, that is over 10 wave-lengths of yellow light, and they point out the desirability of extending the theory to larger spheres.

The calculations referred to related to the particular case where the (relative) refractive index of the spherical obstacles is 1·5. This value was chosen in order to bring out the peculiar polarisation phenomena observed in the diffracted light at angles in the neighbourhood of 90°, and as not inappropriate to experiments upon particles of high index suspended in water. I remarked that the extension of the calculations to greater particles would be of interest, but that the arithmetical work would rapidly become heavy.

There is, however, another particular case of a more tractable character, viz., when the relative refractive index is *small* *; and although it may not be the one we should prefer, its discussion is of interest and would be expected

* [1914. It would have been in better accordance with usage to have said "of Relative Index differing little from Unity."]

† *Roy. Soc. Proc.* A, Vol. LXXXIX. p. 370 (1913).

‡ *Roy. Soc. Proc.* A, Vol. LXXXIV. p. 25 (1910); *Scientific Papers*, Vol. v. p. 547.

to throw some light upon the general course of the phenomenon. It has already been treated up to a certain point, both in the paper cited and the earlier one * in which experiments upon precipitated sulphur were first described. It is now proposed to develop the matter further.

The specific inductive capacity of the general medium being unity, that of the sphere of radius R is supposed to be K, where $K-1$ is very small. Denoting electric displacements by f, g, h, the primary wave is taken to be

$$h_0 = e^{int}\, e^{ikx}, \quad \text{..............................(1)}$$

so that the direction of propagation is along x (negatively), and that of vibration parallel to z. The electric displacements (f_1, g_1, h_1) in the scattered wave, so far as they depend upon the first power of $(K-1)$, have at a great distance the values

$$f_1,\ g_1,\ h_1 = \frac{k^2 P}{4\pi r}\left(\frac{\alpha\gamma}{r^2},\ \frac{\beta\gamma}{r^2},\ -\frac{\alpha^2+\beta^2}{r^2}\right), \quad \text{..............(2)}$$

in which
$$P = -(K-1)\,.\,e^{int}\iiint e^{ik(x-r)}\,dx\,dy\,dz. \quad \text{..................(3)}$$

In these equations r denotes the distance between the point (α, β, γ) where the disturbance is required to be estimated, and the element of volume $(dx\,dy\,dz)$ of the obstacle. The centre of the sphere R will be taken as the origin of coordinates. It is evident that, so far as the secondary ray is concerned, P depends only upon the angle (χ) which this ray makes with the primary ray. We will suppose that $\chi = 0$ in the direction backwards along the primary ray, and that $\chi = \pi$ along the primary ray continued. The integral in (3) may then be found in the form

$$\frac{2\pi R^2 e^{-ikr}}{k\cos\frac{1}{2}\chi}\int_0^{\frac{1}{2}\pi} J_1(2kR\cos\tfrac{1}{2}\chi\,.\,\cos\phi)\cos^2\phi\,d\phi, \quad \text{...........(4)}$$

r now denoting the distance of the point of observation from the centre of the sphere. Expanding the Bessel's function, we get

$$P = -\frac{4\pi R^3(K-1)\,e^{i(nt-kr)}}{3}\left\{1 - \frac{m^2}{2\,.\,5} + \frac{m^4}{2\,.\,4\,.\,5\,.\,7} - \frac{m^6}{2\,.\,4\,.\,6\,.\,5\,.\,7\,.\,9} + \frac{m^8}{2\,.\,4\,.\,6\,.\,8\,.\,5\,.\,7\,.\,9\,.\,11} - \ldots\right\}, \quad \text{.........(5)}$$

in which m is written for $2kR\cos\frac{1}{2}\chi$. It is to be observed that in this solution there is no limitation upon the value of R if $(K-1)^2$ is neglected absolutely. In practice it will suffice that $(K-1)R/\lambda$ be small, λ (equal to $2\pi/k$) being the wave-length.

* *Phil. Mag.* Vol. XII. p. 81 (1881); *Scientific Papers*, Vol. I. p. 518.

These are the formulæ previously given. I had not then noticed that the integral in (4) can be expressed in terms of circular functions. By a general theorem due to Hobson *

$$\int_0^{\frac{1}{2}\pi} J_1(m\cos\phi)\cos^2\phi\, d\phi = \sqrt{\left(\frac{\pi}{2m}\right)} J_{\frac{3}{2}}(m) = \frac{\sin m}{m^2} - \frac{\cos m}{m}, \quad\text{......(6)}$$

so that
$$P = -(K-1) \,.\, 4\pi R^3 \,.\, e^{i(nt-kr)}\left(\frac{\sin m}{m^3} - \frac{\cos m}{m^2}\right), \quad\text{............(7)}$$

in agreement with (5). The secondary disturbance vanishes with P, viz., when $\tan m = m$, or

$$m = 2kR\cos\tfrac{1}{2}\chi = \pi\,(1\cdot4303,\ 2\cdot4590,\ 3\cdot4709,\ 4\cdot4774,\ 5\cdot4818,\ \text{etc.})\dagger. \quad\text{...(8)}$$

The smallest value of kR for which P vanishes occurs when $\chi = 0$, *i.e.* in the direction *backwards* along the primary ray. In terms of λ the diameter is

$$2R = 0\cdot715\lambda. \quad\text{..............................(9)}$$

In directions nearly along the primary ray *forwards*, $\cos\frac{1}{2}\chi$ is small, and evanescence of P requires much larger ratios of R to λ. As was formerly fully discussed, the secondary disturbance vanishes, independently of P, in the direction of primary vibration ($\alpha = 0$, $\beta = 0$).

In general, the intensity of the secondary disturbance is given by

$$f_1^2 + g_1^2 + h_1^2 = \left(\frac{k^2 P_0}{4\pi r}\right)^2\left(1 - \frac{\gamma^2}{r^2}\right), \quad\text{.................(10)}$$

in which P_0 denotes P with the factor $e^{i(nt-kr)}$ omitted, and is a function of χ, the angle between the secondary ray and the axis of x. If we take polar coordinates (χ, ϕ) round the axis of x,

$$1 - \frac{\gamma^2}{r^2} = 1 - \sin^2\chi\,\cos^2\phi; \quad\text{.......................(11)}$$

and the intensity at distance r and direction (χ, ϕ) may be expressed in terms of these quantities. In order to find the effect upon the transmitted light, we have to integrate (10) over the whole surface of the sphere r. Thus

$$r^2\iint \sin\chi\, d\chi\, d\phi\,(f_1^2 + g_1^2 + h_1^2) = \pi\int_0^\pi \sin\chi\, d\chi\left(\frac{k^2 P_0}{4\pi}\right)^2(1+\cos^2\chi)$$

$$= \pi k^4 (K-1)^2 R^6 \int_0^\pi \sin\chi\, d\chi\,(1+\cos^2\chi)\frac{(\sin m - m\cos m)^2}{m^6}$$

$$= \tfrac{1}{2}\pi k^2 R^4 (K-1)^2 \int_0^{2kR} \frac{dm}{m^5}\left\{2 - \frac{m^2}{k^2R^2} + \frac{m^4}{4k^4R^4}\right\}$$

$$\times\{1 + m^2 + (m^2-1)\cos 2m - 2m\sin 2m\}. \quad\text{......(12)}$$

* *Lond. Math. Soc. Proc.* Vol. xxv. p. 71 (1893).

† See *Theory of Sound*, Vol. II. § 207.

The integral may be expressed by means of functions regarded as known. Thus on integration by parts

$$\int_0^m \{1+m^2+(m^2-1)\cos 2m - 2m\sin 2m\}\frac{dm}{m^5}$$

$$= -\frac{1-\cos 2m}{4m^4}+\frac{\sin 2m}{2m^3}-\frac{1}{2m^2}+\frac{1}{2},$$

$$\int_0^m \{1+m^2+(m^2-1)\cos 2m - 2m\sin 2m\}\frac{dm}{m^3}$$

$$= -\frac{1}{2m^2}+\int_0^m \frac{1-\cos 2m}{m}\,dm+\frac{\cos 2m}{2m^2}+\frac{\sin 2m}{m}-1,$$

$$\int_0^m \{1+m^2+(m^2-1)\cos 2m - 2m\sin 2m\}\frac{dm}{m}$$

$$= \int_0^m \frac{1-\cos 2m}{m}\,dm+\frac{m^2}{2}+\frac{m\sin 2m}{2}+\frac{5\cos 2m}{4}-\frac{5}{4}.$$

Accordingly, if m now stand for $2kR$, we get

$$r^2\iint \sin\chi\,d\chi\,d\phi\,(f_1^2+g_1^2+h_1^2)=\frac{\pi R^2(K-1)^2}{8}\left\{-\frac{7(1-\cos 2m)}{2m^2}\right.$$

$$\left.-\frac{\sin 2m}{m}+5+m^2+\left(\frac{4}{m^2}-4\right)\int_0^m \frac{1-\cos 2m}{m}\,dm\right\}. \quad\text{......(13)}$$

If m is small, the $\{\ \}$ in (13) reduces to

$$0+0\times m^2+\tfrac{4}{27}m^4,$$

so that ultimately

$$(13)=\tfrac{8}{27}\pi k^4R^6(K-1)^2, \quad\text{........................(14)}$$

in agreement with the result which may be obtained more simply from (5). If we include another term, we get

$$(13)=\tfrac{8}{27}\pi k^4R^6(K-1)^2\left(1-\frac{2k^2R^2}{5}\right). \quad\text{...............(15)}$$

As regards the definite integral, still written as such, in (13), we have

$$\int_0^m \frac{1-\cos 2m}{m}\,dm=\int_0^{2m}\left\{\frac{x}{1.2}-\frac{x^3}{4\,!}+\frac{x^5}{6\,!}-\dots\right\}dx=\gamma+\log(2m)-\mathrm{Ci}(2m), \quad(16)$$

where γ is Euler's constant (0·5772156) and Ci is the cosine-integral, defined by

$$\mathrm{Ci}\,(x)=\int_\infty^x \frac{\cos u}{u}\,du. \quad\text{..........................(17)}$$

As in (16), when x is moderate, we may use

$$\mathrm{Ci}\,(x)=\gamma+\log x-\tfrac{1}{2}\frac{x^2}{1.2}+\tfrac{1}{4}\frac{x^4}{1.2.3.4}-\dots, \quad\text{...........(18)}$$

which is always convergent. When x is great, we have the semi-convergent series

$$\mathrm{Ci}\,(x) = \sin x \left\{\frac{1}{x} - \frac{1\,.\,2}{x^3} + \frac{1\,.\,2\,.\,3\,.\,4}{x^5} - \ldots\right\}$$
$$- \cos x \left\{\frac{1}{x^2} - \frac{1\,.\,2\,.\,3}{x^4} + \frac{1\,.\,2\,.\,3\,.\,4\,.\,5}{x^6} - \ldots\right\} \ldots\ldots\ldots (19)$$

Fairly complete tables of $\mathrm{Ci}\,(x)$, as well as of related integrals, have been given by Glaisher*.

When m is large, $\mathrm{Ci}\,(2m)$ tends to vanish, so that ultimately

$$\int_0^m \frac{1-\cos 2m}{m}\,dm = \gamma + \log(2m).$$

Hence, when kR is large, (13) tends to the form

$$(13) = \tfrac{1}{2}\pi k^2 R^4 (K-1)^2. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Glaisher's Table XII gives the maxima and minima values of the cosine integral, which occur when the argument is an odd multiple of $\frac{1}{2}\pi$. Thus

n	$\mathrm{Ci}\,(n\pi/2)$	n	$\mathrm{Ci}\,(n\pi/2)$
1	+0·4720007	7	0·0895640
3	−0·1984076	9	+0·0700653
5	+0·1237723	11	−0·0575011

These values allow us to calculate the { } in (13), viz.,

$$-\frac{7(1-\cos 2m)}{2m^2} - \frac{\sin 2m}{m} + 5 + m^2 + \left(\frac{4}{m^2} - 4\right)[\gamma + \log 2m - \mathrm{Ci}\,(2m)], \quad (21)$$

when $2m = n\pi/2$, and n is an odd integer. In this case $\cos 2m = 0$ and $\sin 2m = \pm 1$, so that (21) reduces to

$$-\frac{56}{n^2\pi^2} \pm \frac{4}{n\pi} + 5 + \frac{n^2\pi^2}{16} + \left(\frac{64}{n^2\pi^2} - 4\right)[\gamma + \log(n\pi/2) - \mathrm{Ci}\,(n\pi/2)]. \quad (22)$$

We find

n	(22)	n	(22)
1	0·0530	7	23·440
3	2·718	9	42·382
5	10·534	11	65·958

* *Phil. Trans.* Vol. CLX. p. 367 (1870).

For values of n much greater, (22) is sufficiently represented by $n^2\pi^2/16$, or m^2 simply. It appears that there is no tendency to a falling-off in the scattering, such as would allow an increased transmission.

In order to make sure that the special choice of values for m has not masked a periodicity, I have calculated also the results when n is even. Here $\sin 2m = 0$ and $\cos 2m = \pm 1$, so that (21) reduces to

$$-\frac{112\,(1 \text{ or } 0)}{n^2\pi^2} + 5 + \frac{n^2\pi^2}{16} + \left(\frac{64}{n^2\pi^2} - 4\right)\{\gamma + \log(n\pi/2) - \mathrm{Ci}\,(n\pi/2)\}. \quad (23)$$

The following are required:

n	$\mathrm{Ci}\,(n\pi/2)$	n	$\mathrm{Ci}\,(n\pi/2)$
2	+0·0737	8	−0·0061
4	−0·0224	10	+0·0040
6	+0·0106		

of which the first is obtained by interpolation from Glaisher's Table VI, and the remainder directly from (19). Thus

n	(23)	n	(23)
2	0·7097	8	32·396
4	6·4077	10	53·477
6	16·156		

The better to exhibit the course of the calculation, the actual values of the several terms of (23) when $n = 10$ may be given. We have

$$-\frac{112}{n^2\pi^2} = -0·11348, \quad \frac{n^2\pi^2}{16} = 61·685,$$

$$4 - \frac{64}{n^2\pi^2} = 4 - 0·06485 = 3·93515,$$

$$\gamma + \log(\pi/2) + \log n - \mathrm{Ci}\,(n\pi/2) = 0·57722 + 0·45158 + 2·30259 - 0·0040$$
$$= 3·3274,$$

so that
$$\left(4 - \frac{64}{n^2\pi^2}\right)\{\gamma + \log(n\pi/2) - \mathrm{Ci}\,(n\pi/2)\} = 13·094.$$

It will be seen that from this onwards the term $n^2\pi^2/16$, viz., m^2, greatly preponderates, and this is the term which leads to the limiting form (20).

The values of $2R/\lambda$ concerned in the above are very moderate. Thus, $n = 10$, making $m = 4\pi R/\lambda = 10\pi/4$, gives $2R/\lambda = 5/4$ only. Neither below

this point, nor beyond it, is there anything but a steady rise in the value of (13) as λ diminishes when R is constant. *A fortiori* is this the case when R increases and λ is constant.

An increase in the light scattered from a single spherical particle implies, of course, a decrease in the light directly transmitted through a suspension containing a given number of particles in the cubic centimetre. The calculation is detailed in my paper "On the Transmission of Light through an Atmosphere containing Small Particles in Suspension*," and need not be repeated. It will be seen that no explanation is here arrived at of the augmentation of transparency at a certain stage observed by Keen and Porter. The discrepancy may perhaps be attributed to the fundamental supposition of the present paper, that the relative index is very small [or rather very near unity], a supposition not realised when sulphur and water are in question. But I confess that I should not have expected so wide a difference, and, indeed, the occurrence of anything special at so great diameters as 10 wave-lengths is surprising.

One other matter may be alluded to. It is not clear from the description that the light observed was truly transmitted in the technical sense. This light was much attenuated—down to only 5 per cent. Is it certain that it contained no sensible component of scattered light, but slightly diverted from its original course? If such admixture occurred, the question would be much complicated.

* *Phil. Mag.* Vol. XLVII. p. 375 (1899); *Scientific Papers*, Vol. IV. p. 397.

382.

SOME CALCULATIONS IN ILLUSTRATION OF FOURIER'S THEOREM.

[*Proceedings of the Royal Society*, A, Vol. xc. pp. 318—323, 1914.]

According to Fourier's theorem a curve whose ordinate is arbitrary over the whole range of abscissæ from $x = -\infty$ to $x = +\infty$ can be compounded of harmonic curves of various wave-lengths. If the original curve contain a discontinuity, infinitely small wave-lengths must be included, but if the discontinuity be eased off, infinitely small wave-lengths may not be necessary. In order to illustrate this question I commenced several years ago calculations relating to a very simple case. These I have recently resumed, and although the results include no novelty of principle they may be worth putting upon record.

The case is that where the ordinate is constant (π) between the limits ± 1 for x and outside those limits vanishes.

In general

$$\phi(x) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{+\infty} dv\, \phi(v) \cos k(v-x). \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Here

$$\int_{-\infty}^{+\infty} dv\, \phi(v) \cos k(v-x) = 2\pi \cos kx \int_0^1 dv \cos kv = 2\pi \cos kx \frac{\sin k}{k}$$

$$= \frac{\pi}{k}\{\sin k(x+1) - \sin k(x-1)\},$$

and

$$\phi(x) = \int_0^\infty \frac{dk}{k}\{\sin k(x+1) - \sin k(x-1)\}. \ldots\ldots\ldots\ldots\ldots(2)$$

As is well known, each of the integrals in (2) is equal to $\pm\frac{1}{2}\pi$; so that, as was required, $\phi(x)$ vanishes outside the limits ± 1 and between those limits takes the value π. It is proposed to consider what values are assumed by $\phi(x)$ when in (2) we omit that part of the range of integration in which k exceeds a finite value k_1.

The integrals in (2) are at once expressible by what is called the *sine-integral*, defined by

$$\operatorname{Si}(\theta) = \int_0^\theta \frac{\sin\theta}{\theta}\, d\theta. \quad \text{.........................(3)}$$

Thus $$\phi(x) = \operatorname{Si} k_1(x+1) - \operatorname{Si} k_1(x-1), \quad \text{...................(4)}$$

and if the sine-integral were thoroughly known there would be scarcely anything more to do. For moderate values of θ the integral may be calculated from an ascending series which is always convergent. For larger values this series becomes useless; we may then fall back upon a descending series of the semi-convergent class, viz.,

$$\operatorname{Si}(\theta) = \frac{\pi}{2} - \cos\theta \left\{\frac{1}{\theta} - \frac{1.2}{\theta^3} + \frac{1.2.3.4}{\theta^5} - \dots\right\}$$
$$- \sin\theta \left\{\frac{1}{\theta^2} - \frac{1.2.3}{\theta^4} + \frac{1.2.3.4.5}{\theta^6} - \dots\right\} \text{..........(5)}$$

Dr Glaisher* has given very complete tables extending from $\theta = 0$ to $\theta = 1$, and also from 1 to 5 at intervals of 0·1. Beyond this point he gives the function for integer values of θ from 5 to 15 inclusive, and afterwards only at intervals of 5 for 20, 25, 30, 35, &c. For my purpose these do not suffice, and I have calculated from (5) the values for the missing integers up to $\theta = 60$. The results are recorded in the Table below. In each case, except those quoted from Glaisher, the last figure is subject to a small error.

For the further calculation, involving merely subtractions, I have selected the special cases $k_1 = 1, 2, 10$. For $k_1 = 1$, we have

$$\phi(x) = \operatorname{Si}(x+1) - \operatorname{Si}(x-1). \quad \text{.....................(6)}$$

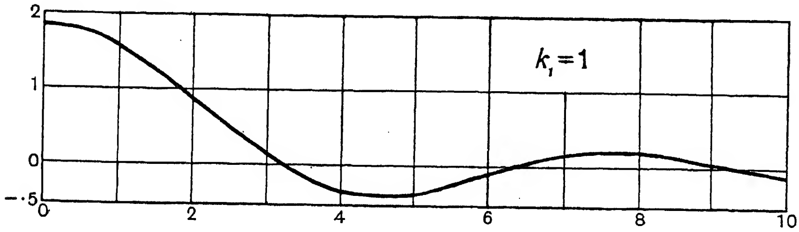
θ	Si (θ)	θ	Si (θ)	θ	Si (θ)	θ	Si (θ)
16	1·63130	28	1·60474	39	1·56334	50	1·55102
17	1·59013	29	1·59731	40	1·58699	51	1·55900
18	1·53662	30	1·56676	41	1·59494	52	1·57357
19	1·51863	31	1·54177	42	1·58083	53	1·58798
20	1·54824	32	1·54424	43	1·55836	54	1·58634
21	1·59490	33	1·57028	44	1·54808	55	1·57072
22	1·61609	34	1·59525	45	1·55871	56	1·55574
23	1·59546	35	1·59692	46	1·57976	57	1·55490
24	1·55474	36	1·57512	47	1·59184	58	1·56845
25	1·53148	37	1·54861	48	1·58445	59	1·58368
26	1·54487	38	1·54549	49	1·56507	60	1·58675
27	1·58029						

In every case $\phi(x)$ is an even function, so that it suffices to consider x positive.

* *Phil. Trans.* Vol. CLX. p. 367 (1870).

$k_1 = 1.$

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
0·0	+1·8922	2·5	+0·5084	6·0	−0·0953
0·5	1·8178	3·0	+0·1528	7·0	+0·1495
1·0	1·6054	3·5	−0·1244	8·0	+0·2104
1·5	1·2854	4·0	−0·2987	9·0	+0·0842
2·0	0·9026	5·0	−0·3335	10·0	−0·0867



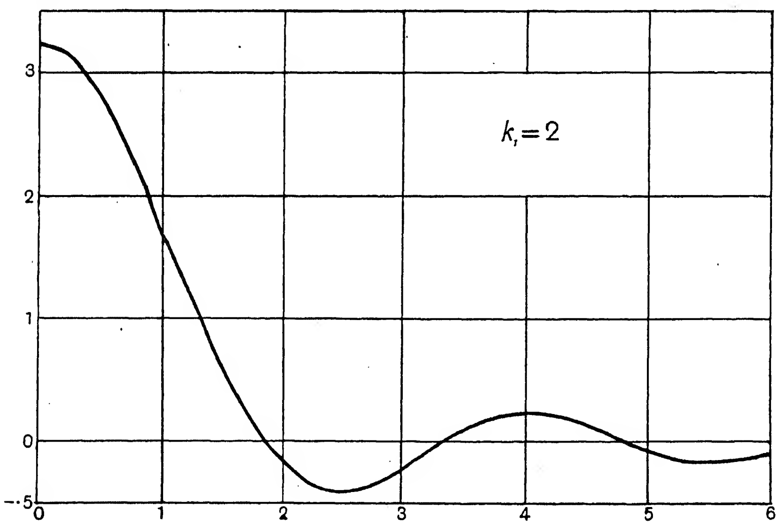
When $k_1 = 2$, $\qquad \phi(x) = \mathrm{Si}(2x+2) - \mathrm{Si}(2x-2)$, ...(7)

and we find

$k_1 = 2.$

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
0·0	+3·2108	0·9	+1·9929	3·0	−0·1840
0·1	3·1934	1·0	1·7582	3·5	+0·1151
0·2	3·1417	1·1	1·5188	4·0	+0·2337
0·3	3·0566	1·2	1·2794	4·5	+0·1237
0·4	2·9401	1·3	1·0443	5·0	−0·0692
0·5	2·7947	1·4	0·8179	5·5	−0·1657
0·6	2·6235	1·5	+0·6038	6·0	−0·1021
0·7	2·4300	2·0	−0·1807		
0·8	2·2184	2·5	−0·3940		

Both for $k_1 = 1$ and for $k_1 = 2$ all that is required for the above values of $\phi(x)$ is given in Glaisher's tables.

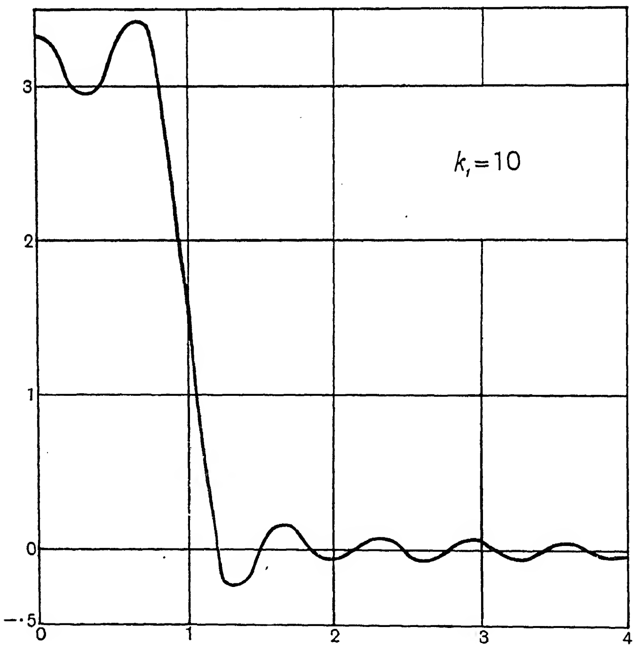


When $k_1 = 10$, $\phi(x) = \text{Si}(10x + 10) - \text{Si}(10x - 10)$. ....(8)

We find

$k_1 = 10$.

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
0·0	+3·3167	1·7	+0·1257	3·4	−0·0067
0·1	3·2433	1·8	+0·0305	3·5	+0·0272
0·2	3·0792	1·9	−0·0677	3·6	+0·0349
0·3	2·9540	2·0	−0·0916	3·7	+0·0115
0·4	2·9809	2·1	−0·0365	3·8	−0·0203
0·5	3·1681	2·2	+0·0393	3·9	−0·0322
0·6	3·3895	2·3	+0·0709	4·0	−0·0151
0·7	3·4388	2·4	+0·0390	4·1	+0·0142
0·8	3·1420	2·5	−0·0213	4·2	+0·0293
0·9	2·4647	2·6	−0·0562	4·3	+0·0178
1·0	1·5482	2·7	−0·0415	4·4	−0·0089
1·1	0·6488	2·8	+0·0089	4·5	−0·0262
1·2	+0·0107	2·9	+0·0447	4·6	−0·0194
1·3	−0·2532	3·0	+0·0387	4·7	+0·0063
1·4	−0·2035	3·1	+0·0000	4·8	+0·0230
1·5	−0·0184	3·2	−0·0353	4·9	+0·0203
1·6	+0·1202	3·3	−0·0371	5·0	−0·0002



The same set of values of Si up to Si(60) would serve also for the calculation of $\phi(x)$ for $k_1 = 20$ and from $x = 0$ to $x = 2$ at intervals of 0·05. It is hardly necessary to set this out in detail.

An inspection of the curves plotted from the above tables shows the approximation towards discontinuity as k_1 increases.

That the curve remains undulatory is a consequence of the sudden stoppage of the integration at $k = k_1$. If we are content with a partial suppression only of the shorter wave-lengths, a much simpler solution is open to us. We have only to introduce into (1) the factor e^{-ak}, where a is positive, and to continue the integration up to $x = \infty$. In place of (2), we have

$$\phi(x) = \int_0^\infty \frac{dk\, e^{-ak}}{k} \{\sin k(x+1) - \sin k(x-1)\} = \tan^{-1}\left(\frac{x+1}{a}\right) - \tan^{-1}\left(\frac{x-1}{a}\right). \quad \text{......(9)}$$

The discontinuous expression corresponds, of course, to $a = 0$. If a is merely small, the discontinuity is eased off. The following are values of $\phi(x)$, calculated from (9) for $a = 1$, 0·5, 0·05:

$a = 1$.

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
0·0	1·571	2·0	0·464	4·0	0·124
0·5	1·446	2·5	0·309	5·0	0·080
1·0	1·107	3·0	0·219	6·0	0·055
1·5	0·727				

$a = 0·5$.

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
0·00	2·214	1·00	1·326	2·00	0·298
0·25	2·173	1·25	0·888	2·50	0·180
0·50	2·111	1·50	0·588	3·00	0·120
0·75	1·756	1·75	0·408	3·50	0·087

$a = 0·05$.

x	$\phi(x)$	x	$\phi(x)$	x	$\phi(x)$
0·00	3·041	0·90	2·652	1·20	0·222
0·20	3·037	0·95	2·331	1·40	0·103
0·40	3·023	1·00	1·546	1·60	0·064
0·60	2·986	1·05	0·761	1·80	0·045
0·80	2·869	1·10	0·440	2·00	0·033

As is evident from the form of (9), $\phi(x)$ falls continuously as x increases whatever may be the value of a.

FURTHER CALCULATIONS CONCERNING THE MOMENTUM OF PROGRESSIVE WAVES.

[*Philosophical Magazine*, Vol. XXVII. pp. 436—440, 1914.]

THE question of the momentum of waves in fluid is of interest and has given rise to some difference of opinion. In a paper published several years ago* I gave an approximate treatment of some problems of this kind. For a fluid moving in one dimension for which the relation between pressure and density is expressed by

$$p = f(\rho), \qquad \qquad (1)$$

it appeared that the momentum of a progressive wave of mean density equal to that of the undisturbed fluid is given by

$$\left\{\frac{\rho_0 f''(\rho_0)}{4a^3} + \frac{1}{2a}\right\} \times \text{total energy}, \qquad \qquad (2)$$

in which ρ_0 is the undisturbed density and a the velocity of propagation. The momentum is reckoned positive when it is in the direction of wave-propagation.

For the "adiabatic" law, viz.:

$$p/p_0 = (\rho/\rho_0)^\gamma, \qquad \qquad (3)$$

$$f'(\rho_0) = \frac{\gamma p_0}{\rho_0} = a^2, \qquad f''(\rho_0) = \frac{p_0\gamma(\gamma-1)}{\rho_0^2}; \qquad \qquad (4)$$

so that

$$\frac{\rho_0 f''(\rho_0)}{4a^3} + \frac{1}{2a} = \frac{\gamma+1}{4a}. \qquad \qquad (5)$$

In the case of Boyle's law we have merely to make $\gamma = 1$ in (5).

For ordinary gases $\gamma > 1$ and the momentum is positive; but the above argument applies to all positive values of γ. If γ be negative, the pressure would increase as the density decreases, and the fluid would be essentially unstable.

* *Phil. Mag.* Vol. x. p. 364 (1905); *Scientific Papers*, Vol. v. p. 265.

However, a slightly modified form of (3) allows the exponent to be negative. If we take

$$p/p_0 = 2 - (\rho/\rho_0)^{-\beta} \quad \text{.............................(6)}$$

with β positive, we get as above

$$f'(\rho_0) = \frac{\beta p_0}{\rho_0} = a^2, \qquad f''(\rho_0) = -\frac{(\beta+1)\, a^2}{\rho_0}, \quad \text{...........(7)}$$

and accordingly

$$\frac{\rho_0 f''(\rho_0)}{4a^3} + \frac{1}{2a} = \frac{1-\beta}{4a}. \quad \text{..........................(8)}$$

If $\beta = 1$, the law of pressure is that under which waves can be propagated without a change of type, and we see that the momentum is zero. In general, the momentum is positive or negative according as β is less or greater than 1.

In the above formula (2) the calculation is approximate only, powers of the disturbance above the *second* being neglected. In the present note it is proposed to determine the sign of the momentum under the laws (3) and (6) more generally and further to extend the calculations to waves in a liquid moving in two dimensions under gravity.

It should be clearly understood that the discussion relates to *progressive* waves. If this restriction be dispensed with, it would always be possible to have a disturbance (limited if we please to a finite length) without momentum, as could be effected very simply by beginning with displacements unaccompanied by velocities. And the disturbance, considered as a whole, can never acquire (or lose) momentum. In order that a wave may be progressive in one direction only, a relation must subsist between the velocity and density at every point. In the case of Boyle's law this relation, first given by De Morgan*, is

$$u = a \log(\rho/\rho_0), \quad \text{.............................(9)}$$

and more generally †

$$u = \int \sqrt{\left(\frac{dp}{d\rho}\right)} \cdot \frac{d\rho}{\rho}. \quad \text{..........................(10)}$$

Wherever this relation is violated, a wave emerges travelling in the negative direction.

For the adiabatic law (3), (10) gives

$$u = \frac{2a}{\gamma - 1}\left\{\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} - 1\right\}, \quad \text{.......................(11)}$$

* Airy, *Phil. Mag.* Vol. xxxiv. p. 402 (1849).

† Earnshaw, *Phil. Trans.* 1859, p. 146.

a being the velocity of infinitely small disturbances, and this reduces to (9) when $\gamma = 1$. Whether γ be greater or less than 1, u is positive when ρ exceeds ρ_0. Similarly if the law of pressure be that expressed in (6),

$$u = \frac{2a}{\beta+1}\left\{1 - \left(\frac{\rho}{\rho_0}\right)^{-\frac{\beta+1}{2}}\right\}. \quad \text{.....................(12)}$$

Since β is positive, values of ρ greater than ρ_0 are here also accompanied by positive values of u.

By definition the momentum of the wave, whose length may be supposed to be limited, is per unit of cross-section

$$\int \rho u \, dx, \quad \text{..................................(13)}$$

the integration extending over the whole length of the wave. If we introduce the value of u given in (11), we get

$$(13) = \frac{2\rho_0 a}{\gamma - 1}\int\left\{\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma+1}{2}} - \frac{\rho}{\rho_0}\right\} dx\,; \quad \text{..................(14)}$$

and the question to be examined is the sign of (14). For brevity we may write unity in place of ρ_0, and we suppose that the wave is such that its mean density is equal to that of the undisturbed fluid, so that $\int \rho \, dx = l$, where l is the length of the wave. If l be divided into n equal parts, then when n is great enough the integral may be represented by the sum

$$\left\{\rho_1^{\frac{\gamma+1}{2}} + \rho_2^{\frac{\gamma+1}{2}} + \rho_3^{\frac{\gamma+1}{2}} + \ldots - \rho_1 - \rho_2 - \ldots\right\}\frac{l}{n}, \quad \text{............(15)}$$

in which all the ρ's are positive. Now it is a proposition in Algebra that

$$\frac{\rho_1^{\frac{\gamma+1}{2}} + \rho_2^{\frac{\gamma+1}{2}} + \ldots}{n} > \left(\frac{\rho_1 + \rho_2 + \ldots}{n}\right)^{\frac{\gamma+1}{2}}$$

when $\frac{1}{2}(\gamma + 1)$ is negative, or positive and greater than unity; but that the reverse holds when $\frac{1}{2}(\gamma+1)$ is positive and less than unity. Of course the inequality becomes an equality when all the n quantities are equal. In the present application the sum of the ρ's is n, and under the adiabatic law (3), γ and $\frac{1}{2}(\gamma+1)$ are positive. Hence (15) is positive or negative according as $\frac{1}{2}(\gamma+1)$ is greater or less than unity, viz., according as γ is greater or less than unity. In either case the momentum represented by (13) is *positive*, and the conclusion is not limited to the supposition of small disturbances.

In like manner if the law of pressure be that expressed in (6), we get from (12)

$$(13) = \frac{2\rho_0 a}{\beta+1}\int\left\{\frac{\rho}{\rho_0} - \left(\frac{\rho}{\rho_0}\right)^{-\frac{\beta-1}{2}}\right\} dx, \quad \text{..................(16)}$$

from which we deduce almost exactly as before that the momentum (13) is positive if β (being positive) is less than 1 and negative if β is greater than 1. If $\beta = 1$, the momentum vanishes. The conclusions formerly obtained on the supposition of small disturbances are thus extended.

We will now discuss the momentum in certain cases of fluid motion under gravity. The simplest is that of *long* waves in a uniform canal. If η be the (small) elevation at any point x measured in the direction of the length of the canal and u the corresponding fluid velocity parallel to x, which is uniform over the section, the dynamical equation is*

$$\frac{du}{dt} = -g\frac{d\eta}{dx}. \quad \text{.............................(17)}$$

As is well known, long waves of small elevation are propagated without change of form. If c be the velocity of propagation, a positive wave may be represented by

$$\eta = F(ct - x), \quad \text{.............................(18)}$$

where F denotes an arbitrary function, and c is related to the depth h_0 according to

$$c^2 = gh_0. \quad \text{...............................(19)}$$

From (17), (18)

$$u = \frac{g\eta}{c} = \sqrt{\left(\frac{g}{h_0}\right)} . \eta \quad \text{..........................(20)}$$

is the relation obtaining between the velocity and elevation at any place in a positive progressive wave of small elevation.

Equation (20), however, does not suffice for our present purpose. We may extend it by the consideration that in a long wave of finite disturbance the elevation and velocity may be taken as relative to the neighbouring parts of the wave. Thus, writing du for u and h for h_0, so that $\eta = dh$, we have

$$du = \sqrt{\left(\frac{g}{h}\right)}\, dh,$$

and on integration

$$u = 2\sqrt{g}\,\{h^{\frac{1}{2}} + C\}.$$

The arbitrary constant of integration is determined by the fact that outside the wave $u = 0$ when $h = h_0$, whence and replacing h by $h_0 + \eta$, we get

$$u = 2\sqrt{g}\,\{\sqrt{(h_0 + \eta)} - \sqrt{h_0}\}, \quad \text{......................(21)}$$

as the generalized form of (20). It is equivalent to a relation given first in another notation by De Morgan†, and it may be regarded as the condition

* Lamb's *Hydrodynamics*, § 168.

† Airy, *Phil. Mag.* Vol. xxxiv. p. 402 (1849).

which must be satisfied if the emergence of a negative wave is to be obviated.

We are now prepared to calculate the momentum. For a wave in which the mean elevation is zero, the momentum corresponding to unit horizontal breadth is

$$\rho\int u(h_0+\eta)\,dx = \tfrac{3}{4}\rho\sqrt{(g/h_0)}\int \eta^2 dx, \quad\ldots\ldots\ldots\ldots\ldots(22)$$

when we omit cubes and higher powers of η. We may write (22) also in the form

$$\text{Momentum} = \frac{3}{4}\,\frac{\text{Total Energy}}{c}, \quad\ldots\ldots\ldots\ldots\ldots(23)$$

c being the velocity of propagation of waves of small elevation.

As in (14), with γ equal to 2, we may prove that the momentum is positive without restriction upon the value of η.

As another example, periodic waves moving on the surface of deep water may also be referred to. The momentum of such waves has been calculated by Lamb*, on the basis of Stokes' second approximation. It appears that the momentum per wave-length and per unit width perpendicular to the plane of motion is

$$\pi\rho a^2 c, \quad\ldots\ldots\ldots\ldots\ldots(24)$$

where c is the velocity of propagation of the waves in question and the wave form is approximately

$$\eta = a\cos\frac{2\pi}{\lambda}(ct-x). \quad\ldots\ldots\ldots\ldots\ldots(25)$$

The forward velocity of the surface layers was remarked by Stokes. For a simple view of the matter reference may be made also to *Phil. Mag.* Vol. I. p. 257 (1876); *Scientific Papers*, Vol. I. p. 263.

* *Hydrodynamics*, § 246.

384.

FLUID MOTIONS.

[*Proc. Roy. Inst.* March, 1914; *Nature*, Vol. XCIII. p. 364, 1914.]

THE subject of this lecture has received the attention of several generations of mathematicians and experimenters. Over a part of the field their labours have been rewarded with a considerable degree of success. In all that concerns small vibrations, whether of air, as in sound, or of water, as in waves and tides, we have a large body of systematized knowledge, though in the case of the tides the question is seriously complicated by the fact that the rotation of the globe is actual and not merely relative to the sun and moon, as well as by the irregular outlines and depths of the various oceans. And even when the disturbance constituting the vibration is not small, some progress has been made, as in the theory of sound waves in one dimension, and of the tidal *bores*, which are such a remarkable feature of certain estuaries and rivers.

The general equations of fluid motion, when friction or viscosity is neglected, were laid down in quite early days by Euler and Lagrange, and in a sense they should contain the whole theory. But, as Whewell remarked, it soon appeared that these equations by themselves take us a surprisingly little way, and much mathematical and physical talent had to be expended before the truths hidden in them could be brought to light and exhibited in a practical shape. What was still more disconcerting, some of the general propositions so arrived at were found to be in flagrant contradiction with observation, even in cases where at first sight it would not seem that viscosity was likely to be important. Thus a solid body, submerged to a sufficient depth, should experience no resistance to its motion through water. On this principle the screw of a submerged boat would be useless, but, on the other hand, its services would not be needed. It is little wonder that practical men should declare that theoretical hydrodynamics has nothing at all to do with real fluids. Later we will return to some of these difficulties, not yet fully surmounted, but for the moment I will call your attention to simple phenomena of which theory can give a satisfactory account.

Considerable simplification attends the supposition that the motion is always the same at the same place—is *steady*, as we say—and fortunately this covers many problems of importance. Consider the flow of water along a pipe whose section varies. If the section were uniform, the pressure would vary along the length only in consequence of friction, which now we are neglecting. In the proposed pipe how will the pressure vary? I will not prophesy as to a Royal Institution audience, but I believe that most unsophisticated people suppose that a contracted place would give rise to an *increased* pressure. As was known to the initiated long ago, nothing can be further from the fact. The experiment is easily tried, either with air or water, so soon as we are provided with the right sort of tube. A suitable shape is shown in fig. 1, but it is rather troublesome to construct in metal.

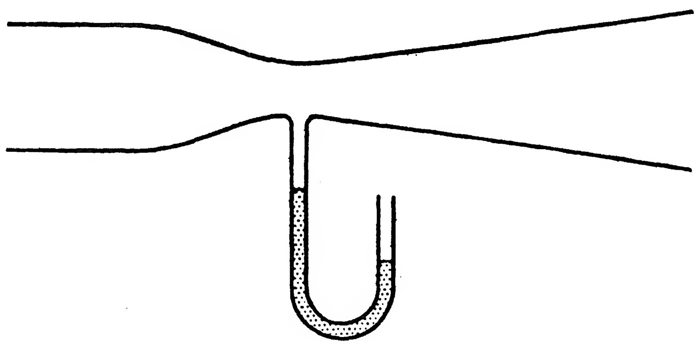
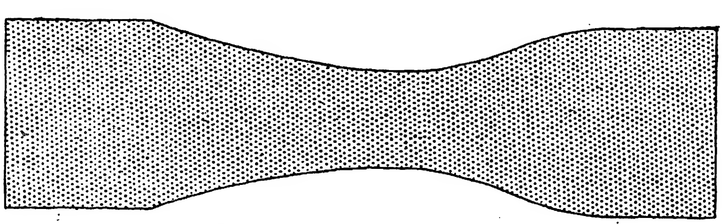


Fig. 1.

W. Froude found paraffin-wax the most convenient material for ship models, and I have followed him in the experiment now shown. A brass tube is filled with candle-wax and bored out to the desired shape, as is easily done with templates of tin plate. When I blow through, a *suction* is developed at the narrows, as is witnessed by the rise of liquid in a manometer connected laterally.

In the laboratory, where dry air from an acoustic bellows or a gas-holder is available, I have employed successfully tubes built up of cardboard, for a circular cross-section is not necessary. Three or more precisely similar pieces, cut for example to the shape shown in fig. 2 and joined together



closely along the edges, give the right kind of tube, and may be made air-tight with pasted paper or with sealing-wax. Perhaps a square section requiring four pieces is best. It is worth while to remark that there is no stretching of the cardboard, each side being merely *bent* in one dimension. A model is before you, and a study of it forms a simple and useful exercise in solid geometry.

Another form of the experiment is perhaps better known, though rather more difficult to think about. A tube (fig. 3) ends in a flange. If I blow through the tube, a card presented to the flange is drawn up pretty closely, instead of being blown away as might be expected. When we consider the

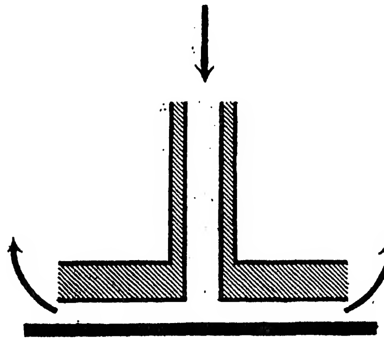


Fig. 3.

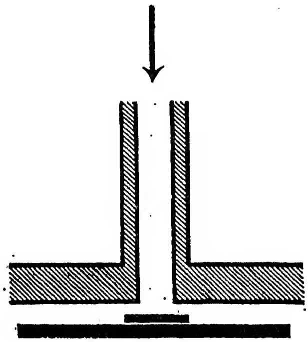


Fig. 4.

matter, we recognize that the channel between the flange and the card through which the air flows after leaving the tube is really an expanding one, and thus that the inner part may fairly be considered as a contracted place. The suction here developed holds the card up.

A slight modification enhances the effect. It is obvious that immediately opposite the tube there will be pressure upon the card and not suction. To neutralize this a sort of cap is provided, attached to the flange, upon which the objectionable pressure is taken (fig. 4). By blowing smartly from the mouth through this little apparatus it is easy to lift and hold up a penny for a short time.

The facts then are plain enough, but what is the explanation? It is really quite simple. In steady motion the quantity of fluid per second passing any section of the tube is everywhere the same. If the fluid be incompressible, and air in these experiments behaves pretty much as if it were, this means that the product of the velocity and area of cross-section is constant, so that at a narrow place the velocity of flow is necessarily increased. And when we enquire how the additional velocity in passing from a wider to a narrower place is to be acquired, we are compelled to recognize that it can only be in consequence of a fall of pressure. The suction at the narrows is the only result consistent with the great principle of conservation of energy;

but it remains rather an inversion of ordinary ideas that we should have to deduce the forces from the motion, rather than the motion from the forces.

The application of the principle is not always quite straightforward. Consider a tube of slightly conical form, open at both ends, and suppose that we direct upon the narrower end a jet of air from a tube having the same (narrower) section (fig. 5). We might expect this jet to enter the

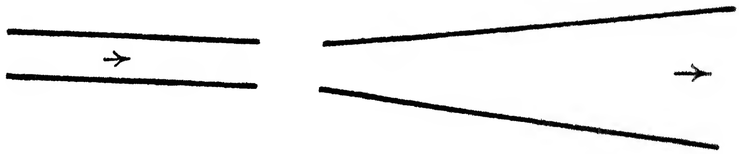


Fig. 5.

conical tube without much complication. But if we examine more closely a difficulty arises. The stream in the conical tube would have different velocities at the two ends, and therefore different pressures. The pressures at the ends could not both be atmospheric. Since at any rate the pressure at the wider delivery end must be very nearly atmospheric, that at the narrower end must be decidedly below that standard. The course of the events at the inlet is not so simple as supposed, and the apparent contradiction is evaded by an inflow of air from outside, in addition to the jet, which assumes at entry a narrower section.

If the space surrounding the free jet is enclosed (fig. 6), suction is there developed and ultimately when the motion has become steady the jet enters the conical tube without contraction. A model shows the effect, and the principle is employed in a well-known laboratory instrument arranged for working off the water-mains.

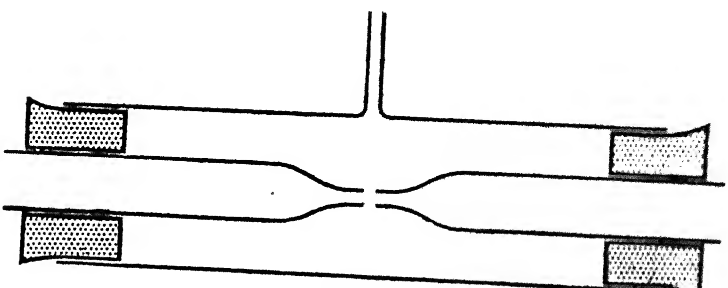


Fig. 6.

I have hitherto dealt with air rather than water, not only because air makes no mess, but also because it is easier to ignore gravitation. But there is another and more difficult question. You will have noticed that in our expanding tubes the section changes only gradually. What happens when the expansion is more sudden—in the extreme case when the diameter of a previously uniform tube suddenly becomes infinite? (fig. 3) without

card. Ordinary experience teaches that in such a case the flow does not follow the walls round the corner, but shoots across as a jet, which for a time preserves its individuality and something like its original section. Since the velocity is not lost, the pressure which would replace it is not developed. It is instructive to compare this case with another, experimented on by Savart* and W. Froude †, in which a free jet is projected through a short cone, or a mere hole in a thin wall, into a vessel under a higher pressure. The apparatus consists of two precisely similar vessels with apertures, in which the fluid (water) may be at different levels (fig. 7, copied from Froude). Savart found that not a single drop of liquid was spilt so long as the pressure in the recipient vessel did not exceed one-sixth of that under which the jet issues. And Froude reports that so long as the head in the discharge cistern is maintained at a moderate height above that in the

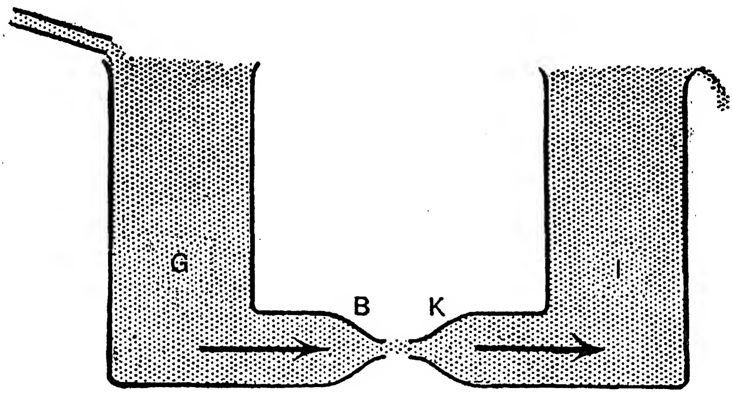


Fig. 7.

recipient cistern, the whole of the stream enters the recipient orifice, and there is "no waste, except the small sprinkling which is occasioned by inexactness of aim, and by want of exact circularity in the orifices." I am disposed to attach more importance to the small spill, at any rate when the conoids are absent or very short. For if there is no spill, the jet (it would seem) might as well be completely enclosed; and then it would propagate itself into the recipient cistern without sudden expansion and consequent recovery of pressure. In fact, the pressure at the narrows would never fall below that of the recipient cistern, and the discharge would be correspondingly lessened. When a decided spill occurs, Froude explains it as due to the retardation by friction of the outer layers which are thus unable to force themselves against the pressure in front.

Evidently it is the behaviour of these outer layers, especially at narrow places, which determines the character of the flow in a large variety of cases.

* *Ann. de Chimie*, Vol. LV. p. 257, 1833.
† *Nature*, Vol. XIII. p. 93, 1875.

They are held back, as Froude pointed out, by friction acting from the walls; but, on the other hand, when they lag, they are pulled forward by layers farther in which still retain their velocity. If the latter prevail, the motion in the end may not be very different from what would occur in the absence of friction; otherwise an entirely altered motion may ensue. The situation as regards the rest of the fluid is much easier when the layers upon which the friction tells most are allowed to escape. This happens in instruments of the injector class, but I have sometimes wondered whether full advantage is taken of it. The long gradually expanding cones are overdone, perhaps, and the friction which they entail must have a bad effect.

Similar considerations enter when we discuss the passage of a solid body through a large mass of fluid otherwise at rest, as in the case of an airship or submarine boat. I say a submarine, because when a ship moves upon the surface of the water the formation of waves constitutes a complication, and one of great importance when the speed is high. In order that the water in its relative motion may close in properly behind, the after-part of the ship must be suitably shaped, fine lines being more necessary at the stern than at the bow, as fish found out before men interested themselves in the problem. In a well-designed ship the whole resistance (apart from wave-making) may be ascribed to *skin friction*, of the same nature as that which is encountered when the ship is replaced by a thin plane moving edgeways.

At the other extreme we may consider the motion of a thin disk or blade flatways through the water. Here the actual motion differs altogether from that prescribed by the classical hydrodynamics, according to which the character of the motion should be the same behind as in front. The liquid refuses to close in behind, and a region of more or less "dead water" is developed, entailing a greatly increased resistance. To meet this Helmholtz, Kirchhoff, and their followers have given calculations in which the fluid behind is supposed to move strictly with the advancing solid, and to be separated from the remainder of the mass by a surface at which a finite slip takes place. Although some difficulties remain, there can be no doubt that this theory constitutes a great advance. But the surface of separation is unstable, and in consequence of fluid friction it soon loses its sharpness, breaking up into more or less periodic eddies, described in some detail by Mallock (fig. 8). It is these eddies which cause the whistling of the wind in trees and the more musical notes of the aeolian harp.

The obstacle to the closing-in of the lines of flow behind the disk is doubtless, as before, the layer of liquid in close proximity to the disk, which at the edge has insufficient velocity for what is required of it. It would be an interesting experiment to try what would be the effect of allowing a small "spill." For this purpose the disk or blade would be made double, with a suction applied to the narrow interspace. Relieved of the slowly

moving layer, the liquid might then be able to close in behind, and success would be witnessed by a greatly diminished resistance.

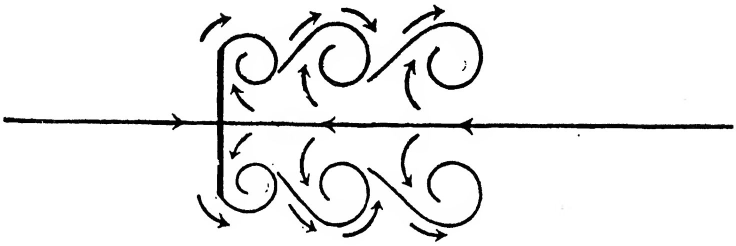


Fig. 8.

When a tolerably fair-shaped body moves through fluid, the relative velocity is greatest at the maximum section of the solid which is the minimum section for the fluid, and consequently the pressure is there least. Thus the water-level is depressed at and near the midship section of an advancing steamer, as is very evident in travelling along a canal. On the same principle may be explained the stability of a ball sustained on a vertical jet as in a well-known toy (shown). If the ball deviate to one side, the jet in bending round the surface develops a suction pulling the ball back. As Mr Lanchester has remarked, the effect is aided by the rotation of the ball. That a convex surface is attracted by a jet playing obliquely upon it was demonstrated by T. Young more than 100 years ago by means of a model, of which a copy is before you (fig. 9).

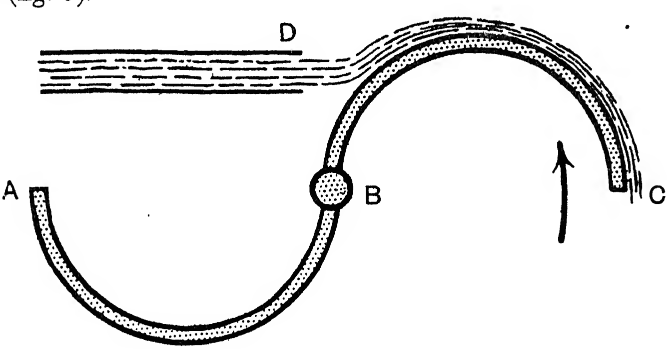


Fig. 9.

A plate, bent into the form *ABC*, turning on centre *B*, is impelled by a stream of air *D* in the direction shown.

It has been impossible in dealing with experiments to keep quite clear of friction, but I wish now for a moment to revert to the ideal fluid of hydrodynamics, in which pressure and inertia alone come into account. The possible motions of such a fluid fall into two great classes—those which do and those which do not involve *rotation*. What exactly is meant by rotation is best explained after the manner of Stokes. If we imagine any spherical

16—2

portion of the fluid in its motion to be suddenly solidified, the resulting solid may be found to be rotating. If so, the original fluid is considered to possess rotation. If a mass of fluid moves irrotationally, no spherical portion would revolve on solidification. The importance of the distinction depends mainly upon the theorem, due to Lagrange and Cauchy, that the irrotational character is permanent, so that any portion of fluid at any time destitute of rotation will always remain so. Under this condition fluid motion is comparatively simple, and has been well studied. Unfortunately many of the results are very unpractical.

As regards the other class of motions, the first great step was taken in 1858, by Helmholtz, who gave the theory of the vortex-ring. In a perfect fluid a vortex-ring has a certain permanence and individuality, which so much impressed Kelvin that he made it the foundation of a speculation as to the nature of matter. To him we owe also many further developments in pure theory.

On the experimental side, the first description of vortex-rings that I have come across is that by W. B. Rogers*, who instances their production during the bursting of bubbles of phosphuretted hydrogen, or the escape of smoke from cannon and from the lips of expert tobacconists. For private observation nothing is simpler than Helmholtz's method of drawing a partially immersed spoon along the surface, for example, of a cup of tea. Here half a ring only is developed, and the places where it meets the surface are shown as dimples, indicative of diminished pressure. The experiment, made on a larger scale, is now projected upon the screen, the surface of the liquid and its motion being made more evident by powder of lycopodium or sulphur scattered over it. In this case the ring is generated by the motion of a half-immersed circular disk, withdrawn after a travel of two or three inches. In a modified experiment the disk is replaced by a circular or semi-circular *aperture* cut in a larger plate, the level of the water coinciding with the horizontal diameter of the aperture. It may be noticed that while the first forward motion of the plate occasions a ring behind, the stoppage of the plate gives rise to a second ring in front. As was observed by Reusch †, the same thing occurs in the more usual method of projecting smoke-rings from a box; but in order to see it the box must be transparent.

In a lecture given here in 1877, Reynolds showed that a Helmholtz ring can push the parent disk before it, so that for a time there appears to be little resistance to its motion.

For an explanation of the origin of these rings we must appeal to friction, for in a perfect fluid no rotation can develop. It is easy to recognize that friction against the wall in which the aperture is perforated, or against the

* *Amer. J. Sci.* Vol. XXVI. p. 246, 1858.

† *Pogg. Ann.* Vol. CX. p. 309, 1860.

face of the disk in the other form of experiment, will start a rotation which, in a viscous fluid, such as air or water actually is, propagates itself to a finite distance inwards. But although a general explanation is easy, many of the details remain obscure.

It is apparent that in dealing with a large and interesting class of fluid motions we cannot go far without including fluid friction, or *viscosity* as it is generally called, in order to distinguish it from the very different sort of friction encountered by solids, unless well lubricated. In order to define it, we may consider the simplest case where fluid is included between two parallel walls, at unit distance apart, which move steadily, each in its own plane, with velocities which differ by unity. On the supposition that the fluid also moves in plane strata, the viscosity is measured by the tangential force per unit of area exercised by each stratum upon its neighbours. When we are concerned with internal motions only, we have to do rather with the so-called "kinematic viscosity," found by dividing the quantity above defined by the density of the fluid. On this system the viscosity of water is much *less* than that of air.

Viscosity varies with temperature; and it is well to remember that the viscosity of air increases while that of water decreases as the temperature rises. Also that the viscosity of water may be greatly increased by admixture with alcohol. I used these methods in 1879 during investigations respecting the influence of viscosity upon the behaviour of such fluid jets as are sensitive to sound and vibration.

Experimentally the simplest case of motion in which viscosity is paramount is the flow of fluid through capillary tubes. The laws of such motion are simple, and were well investigated by Poiseuille. This is the method employed in practice to determine viscosities. The apparatus before you is arranged to show the diminution of viscosity with rising temperature. In the cold the flow of water through the capillary tube is slow, and it requires sixty seconds to fill a small measuring vessel. When, however, the tube is heated by passing steam through the jacket surrounding it, the flow under the same head is much increased, and the measure is filled in twenty-six seconds. Another case of great practical importance, where viscosity is the leading consideration, relates to lubrication. In admirably conducted experiments Tower showed that the solid surfaces moving over one another should be separated by a complete film of oil, and that when this is attended to there is no wear. On this basis a fairly complete theory of lubrication has been developed, mainly by O. Reynolds. But the capillary nature of the fluid also enters to some extent, and it is not yet certain that the whole character of a lubricant can be expressed even in terms of both surface tension and viscosity.

It appears that in the extreme cases, when viscosity can be neglected and again when it is paramount, we are able to give a pretty good account of

what passes. It is in the intermediate region, where both inertia and viscosity are of influence, that the difficulty is greatest. But even here we are not wholly without guidance. There is a general law, called the law of dynamical similarity, which is often of great service. In the past this law has been unaccountably neglected, and not only in the present field. It allows us to infer what will happen upon one scale of operations from what has been observed at another. On the present occasion I must limit myself to viscous fluids, for which the law of similarity was laid down in all its completeness by Stokes as long ago as 1850. It appears that similar motions may take place provided a certain condition be satisfied, viz. that the product of the linear dimension and the velocity, divided by the kinematic viscosity of the fluid, remain unchanged. Geometrical similarity is presupposed. An example will make this clearer. If we are dealing with a single fluid, say air under given conditions, the kinematic viscosity remains of course the same. When a solid sphere moves uniformly through air, the character of the motion of the fluid round it may depend upon the size of the sphere and upon the velocity with which it travels. But we may infer that the motions remain *similar*, if only the product of diameter and velocity be given. Thus, if we know the motion for a particular diameter and velocity of the sphere, we can infer what it will be when the velocity is halved and the diameter doubled. The fluid velocities also will everywhere be halved at the *corresponding* places. M. Eiffel found that for any sphere there is a velocity which may be regarded as critical, *i.e.* a velocity at which the law of resistance changes its character somewhat suddenly. It follows from the rule that these critical velocities should be inversely proportional to the diameters of the spheres, a conclusion in pretty good agreement with M. Eiffel's observations*. But the principle is at least equally important in effecting a comparison between different fluids. If we know what happens on a certain scale and at a certain velocity in *water*, we can infer what will happen in *air* on any other scale, provided the velocity is chosen suitably. It is assumed here that the compressibility of the air does not come into account, an assumption which is admissible so long as the velocities are small in comparison with that of sound.

But although the principle of similarity is well established on the theoretical side and has met with some confirmation in experiment, there has been much hesitation in applying it, due perhaps to certain discrepancies with observation which stand recorded. And there is another reason. It is rather difficult to understand how viscosity can play so large a part as it seems to do, especially when we introduce numbers, which make it appear that the viscosity of air, or water, is very small in relation to the other data occurring in practice. In order to remove these doubts it is very desirable to experiment with different viscosities, but this is not easy to do on a

* *Comptes Rendus*, Dec. 30, 1912, Jan. 13, 1913. [This volume, p. 136.]

moderately large scale, as in the wind channels used for aeronautical purposes. I am therefore desirous of bringing before you some observations that I have recently made with very simple apparatus.

When liquid flows from one reservoir to another through a channel in which there is a contracted place, we can compare what we may call the *head* or driving pressure, *i.e.* the difference of the pressures in the two reservoirs, with the *suction*, *i.e.* the difference between the pressure in the recipient vessel and that lesser pressure to be found at the narrow place. The ratio of head to suction is a purely numerical quantity, and according to the principle of similarity it should for a given channel remain unchanged, provided the velocity be taken proportional to the kinematic viscosity of the fluid. The use of the same material channel throughout has the advantage that no question can arise as to geometrical similarity, which in principle should extend to any roughnesses upon the surface, while the necessary changes of velocity are easily attained by altering the head and those of viscosity by altering the temperature.

The apparatus consisted of two aspirator bottles (fig. 10) containing water and connected below by a passage bored in a cylinder of lead, 7 cm.

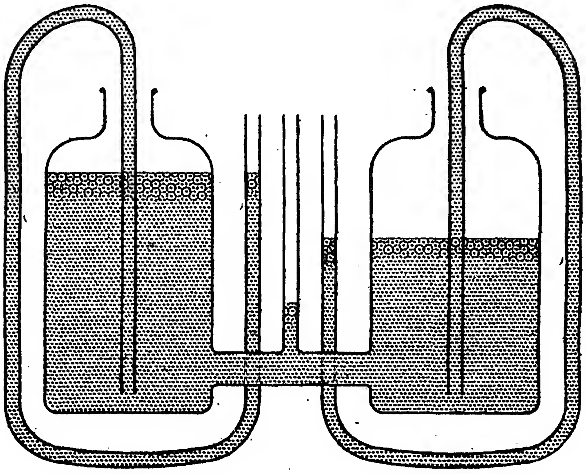


Fig. 10.

long, fitted water-tight with rubber corks. The form of channel actually employed is shown in fig. 11. On the up-stream side it contracts pretty suddenly from full bore (8 mm.) to the narrowest place, where the diameter is 2·75 mm. On the down-stream side the expansion takes place in four or five steps, corresponding to the drills available. It had at first been intended to use a smooth curve, but preliminary trials showed that this was unnecessary, and the expansion by steps has the advantage of bringing before the mind the dragging action of the jets upon the thin layers of fluid

between them and the walls. The three pressures concerned are indicated on manometer tubes as shown, and the two differences of level representing head and suction can be taken off with compasses and referred to a millimetre scale. In starting an observation the water is drawn up in the discharge vessel, as far as may be required, with the aid of an air-pump. The rubber cork at the top of the discharge vessel necessary for this purpose is not shown.

As the head falls during the flow of the water, the ratio of head to suction increases. For most of the observations I contented myself with recording the head for which the ratio of head to suction was exactly 2 : 1, as indicated by proportional compasses. Thus on January 23, when the temperature of the water was 9° C., the 2 : 1 ratio occurred on four trials at 120, 130, 123, 126, mean 125 mm. head. The temperature was then raised with precaution by pouring in warm water with passages backwards and forwards. The occurrence of the 2 : 1 ratio was now much retarded, the mean head being only 35 mm., corresponding to a mean temperature of 37° C. The ratio of

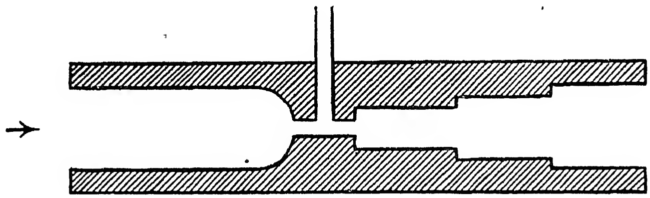


Fig. 11.

head to suction is thus dependent upon the head or velocity, but when the velocity is altered the original ratio may be recovered if at the same time we make a suitable alteration of viscosity.

And the required alteration of viscosity is about what might have been expected. From Landolt's tables I find that for 9° C. the viscosity of water is ·01368, while for 37° C. it is ·00704. The ratio of viscosities is accordingly 1·943. The ratio of heads is 125 : 35. The ratio of *velocities* is the square-root of this or 1·890, in sufficiently good agreement with the ratio of viscosities.

In some other trials the ratio of velocities *exceeded* a little the ratio of viscosities. It is not pretended that the method would be an accurate one for the comparison of viscosities. The change in the ratio of head to suction is rather slow, and the measurement is usually somewhat prejudiced by unsteadiness in the suction manometer. Possibly better results would be obtained in more elaborate observations by several persons, the head and suction being recorded separately and referred to a time scale so as to facilitate interpolation. But as they stand the results suffice for my purpose, showing directly and conclusively the influence of viscosity as compensating a change in the velocity.

In conclusion, I must touch briefly upon a part of the subject where theory is still at fault, and I will limit myself to the simplest case of all—the uniform shearing motion of a viscous fluid between two parallel walls, one of which is at rest, while the other moves tangentially with uniform velocity. It is easy to prove that a uniform shearing motion of the fluid satisfies the dynamical equations, but the question remains: Is this motion stable? Does a small departure from the simple motion tend of itself to die out? In the case where the viscosity is relatively great, observation suggests an affirmative answer; and O. Reynolds, whose illness and comparatively early death were so great a loss to science, was able to deduce the same conclusion from theory. Reynolds' method has been improved, more especially by Professor Orr of Dublin. The simple motion is thoroughly stable if the viscosity exceed a certain specified value relative to the velocity of the moving plane and the distance between the planes; while if the viscosity is less than this, it is possible to propose a kind of departure from the original motion which will increase *for a time*. It is on this side of the question that there is a deficiency. When the viscosity is very small, observation appears to show that the simple motion is unstable, and we ought to be able to derive this result from theory. But even if we omit viscosity altogether, it does not appear possible to prove instability *à priori*, at least so long as we regard the walls as mathematically plane. We must confess that at the present we are unable to give a satisfactory account of skin-friction, in order to overcome which millions of horse-power are expended in our ships. Even in the older subjects there are plenty of problems left!

ON THE THEORY OF LONG WAVES AND BORES.

[*Proceedings of the Royal Society*, A, Vol. xc. pp. 324—328, 1914.]

IN the theory of long waves in two dimensions, which we may suppose to be reduced to a "steady" motion, it is assumed that the length is so great in proportion to the depth of the water that the velocity in a vertical direction can be neglected, and that the horizontal velocity is uniform across each section of the canal. This, it should be observed, is perfectly distinct from any supposition as to the height of the wave. If l be the undisturbed depth, and h the elevation of the water at any point of the wave, u_0, u the velocities corresponding to l, $l+h$ respectively, we have, as the equation of continuity,

$$u = \frac{lu_0}{l+h}. \quad \text{.................................(1)}$$

By the principles of hydrodynamics, the increase of pressure due to retardation will be

$$\tfrac{1}{2}\rho\,(u_0^2 - u^2) = \frac{\rho u_0^2}{2}\cdot\frac{2lh + h^2}{(l+h)^2}. \quad \text{.....................(2)}$$

On the other hand, the loss of pressure (at the surface) due to height will be $g\rho h$; and therefore the total gain of pressure over the undisturbed parts is

$$\left(\frac{\rho u_0^2}{l}\cdot\frac{1 + h/2l}{(1+h/l)^2} - g\rho\right) h. \quad \text{........................(3)}$$

If, now, the ratio h/l be very small, the coefficient of h becomes

$$\rho\,(u_0^2/l - g), \quad \text{.................................(4)}$$

and we conclude that the condition of a free surface is satisfied, provided $u_0^2 = gl$. This determines the rate of flow u_0, in order that a stationary wave may be possible, and gives, of course, at the same time the velocity of a wave in still water.

Unless h^2 can be neglected, it is impossible to satisfy the condition of a free surface for a stationary long wave—which is the same as saying that it is impossible for a long wave of finite height to be propagated in still water without change of type.

Although a constant gravity is not adequate to compensate the changes of pressure due to acceleration and retardation in a long wave of finite height, it is evident that complete compensation is attainable if gravity be made a suitable function of height; and it is worth while to enquire what the law of force must be in order that long waves of unlimited height may travel with type unchanged. If f be the force at height h, the condition of constant surface pressure is

$$\tfrac{1}{2}\rho u_0^2 \left\{1 - \frac{l^2}{(l+h)^2}\right\} = \rho \int_0^h f\, dh\,; \qquad \text{.....................(5)}$$

whence

$$f = -\frac{u_0^2}{2} \cdot \frac{d}{dh} \frac{l^2}{(l+h)^2} = u_0^2 \frac{l^2}{(l+h)^3}, \qquad \text{..................(6)}$$

which shows that the force must vary inversely as the cube of the distance from the bottom of the canal. Under this law the waves may be of any height, and they will be propagated unchanged with the velocity $\sqrt{(f_1 l)}$, where f_1 is the force at the undisturbed level*.

It may be remarked that we are concerned only with the values of f at water-levels which actually occur. A change in f below the lowest water-level would have no effect upon the motion, and thus no difficulty arises from the law of inverse cube making the force infinite at the bottom of the canal.

When a wave is limited in length, we may speak of its velocity relatively to the undisturbed water lying beyond it on the two sides, and it is implied that the uniform levels on the two sides are the same. But the theory of long waves is not thus limited, and we may apply it to the case where the uniform levels on the two sides of the variable region are different, as, for example, to *bores*. This is a problem which I considered briefly on a former occasion†, when it appeared that the condition of conservation of energy could not be satisfied with a constant gravity. But in the calculation of the loss of energy a term was omitted, rendering the result erroneous, although the general conclusions are not affected. The error became apparent in applying the method to the case above considered of a gravity varying as the inverse cube of the depth. But, before proceeding to the calculation of energy, it may be well to give the generalised form of the relation between velocity and height which must be satisfied in a *progressive* wave‡, whether or not the type be permanent.

* *Phil. Mag.* Vol. I. p. 257 (1876); *Scientific Papers*, Vol. I. p. 254.
† *Roy. Soc. Proc.* A, Vol. LXXXI. p. 448 (1908); *Scientific Papers*, Vol. V. p. 495.
‡ Compare *Scientific Papers*, Vol. I. p. 253 (1899).

In a small positive progressive wave, the relation between the particle-velocity u at any point (now reckoned relatively to the parts outside the wave) and the elevation h is

$$u = \sqrt{(f/l)} \, . \, h. \qquad (7)$$

If this relation be violated anywhere, a wave will emerge, travelling in the negative direction. In applying (7) to a wave of finite height, the appropriate form of (7) is

$$du = \sqrt{\left(\frac{f}{l+h}\right)} \, dh, \qquad (8)$$

where f is a known function of $l+h$, or on integration

$$u = \int_0^h \sqrt{\left(\frac{f}{l+h}\right)} \, dh. \qquad (9)$$

To this particle-velocity is to be added the wave-velocity

$$\sqrt{\{(l+h)f\}}, \qquad (10)$$

making altogether for the velocity of, *e.g.*, the crest of a wave relative to still water

$$\int_0^h \sqrt{\left(\frac{f}{l+h}\right)} \, dh + \sqrt{\{(l+h)f\}}. \qquad (11)$$

Thus if f be constant, say g, (9) gives De Morgan's formula

$$u = 2\sqrt{g}\,\{(l+h)^{\frac{1}{2}} - l^{\frac{1}{2}}\}, \qquad (12)$$

and (11) becomes

$$3\sqrt{g}\sqrt{(l+h)} - 2\sqrt{(gl)}. \qquad (13)$$

If, again,

$$f = \frac{f_1 l^3}{(l+h)^3}, \qquad (14)$$

(11) gives as the velocity of a crest

$$\frac{f_1^{\frac{1}{2}} l^{\frac{1}{2}} h}{l+h} + \frac{f_1^{\frac{1}{2}} l^{\frac{3}{2}}}{l+h} = \sqrt{(f_1 l)}, \qquad (15)$$

which is independent of h, thus confirming what was found before for this law of force.

As regards the question of a bore, we consider it as the transition from a uniform velocity u and depth l to a uniform velocity u' and depth l', l' being greater than l. The first relation between these four quantities is that given by continuity, viz.,

$$lu = l'u'. \qquad (16)$$

The second relation arises from a consideration of momentum. It may be convenient to take first the usual case of a constant gravity g. The mean pressures at the two sections are $\frac{1}{2}gl$, $\frac{1}{2}gl'$, and thus the equation of momentum is

$$lu\,(u - u') = \tfrac{1}{2}g\,(l'^2 - l^2). \qquad (17)$$

By these equations u and u' are determined in terms of l, l':

$$u^2 = \tfrac{1}{2} g (l + l') \,.\, l'/l, \qquad u'^2 = \tfrac{1}{2} g (l + l') \,.\, l/l'. \qquad \text{............(18)}$$

We have now to consider the question of energy. The difference of work done by the pressure at the two ends (reckoned per unit of time and per unit of breadth) is $lu(\tfrac{1}{2}gl - \tfrac{1}{2}gl')$. And the difference between the *kinetic* energies entering and leaving the region is $lu(\tfrac{1}{2}u^2 - \tfrac{1}{2}u'^2)$, the density being taken as unity. But this is not all. The *potential* energies of the liquid leaving and entering the region are different. The centre of gravity rises through a height $\tfrac{1}{2}(l' - l)$, and the gain of potential energy is therefore $lu \,.\, \tfrac{1}{2} g (l' - l)$. The whole *loss* of energy is accordingly

$$lu \{\tfrac{1}{2}gl - \tfrac{1}{2}gl' + \tfrac{1}{2}u^2 - \tfrac{1}{2}u'^2 - \tfrac{1}{2}g(l' - l)\} = lu \left\{gl - gl' + \tfrac{1}{4} g (l + l') \left(\frac{l'}{l} + \frac{l}{l'}\right)\right\}$$

$$= lu \,.\, \frac{g(l' - l)^3}{4ll'}. \qquad \text{..........................(19)}$$

This is much smaller than the value formerly given, but it remains of the same sign. "That there should be a loss of energy constitutes no difficulty, at least in the presence of viscosity; but the impossibility of a gain of energy shows that the motions here contemplated cannot be reversed."

We now suppose that the constant gravity is replaced by a force f, which is a function of y, the distance from the bottom. The pressures p, p' at the two sections are also functions of y, such that

$$p = \int_y^l f \, dy, \qquad p' = \int_y^{l'} f \, dy. \qquad \text{....................(20)}$$

The equation of momentum replacing (17) is now

$$lu(u - u') = \int_0^{l'} p' \, dy - \int_0^l p \, dy = \Big[p'y\Big]_0^{l'} - \Big[py\Big]_0^l - \int_0^{l'} y \frac{dp'}{dy} dy + \int_0^l y \frac{dp}{dy} dy$$

$$= \int_0^{l'} yf \, dy - \int_0^l yf \, dy = \int_l^{l'} yf \, dy, \qquad \text{..............................(21)}$$

the integrated terms vanishing at the limits. This includes, of course, all special cases, such as $f =$ constant, or $f \propto y^{-3}$.

As regards the reckoning of energy, the first two terms on the left of (19) are replaced by

$$lu \left\{\frac{1}{l} \int_0^l p \, dy - \frac{1}{l'} \int_0^{l'} p' dy\right\}. \qquad \text{....................(22)}$$

The third and fourth terms representing kinetic energy remain as before. For the potential energy we have to consider that a length u and depth l is converted into a length u' and depth l'. If we reckon from the bottom, the potential energy is in the first case

$$u \int_0^l dy \int_0^y f \, dy,$$

in which

$$\int_0^y f\,dy = \int_0^l f\,dy - \int_y^l f\,dy = p_0 - p,$$

p_0 denoting the pressure at the bottom, so that the potential energy is

$$ul\left\{p_0 - \frac{1}{l}\int_0^l p\,dy\right\}.$$

The difference of potential energies, corresponding to the fifth and sixth terms of (19), is thus

$$lu\left\{p_0 - p_0' - \frac{1}{l}\int_0^l p\,dy + \frac{1}{l'}\int_0^{l'} p'\,dy\right\}. \qquad \text{.............}(23)$$

The integrals in (23) compensate those of (22), and we have finally as the loss of energy

$$lu\left\{p_0 - p_0' + \tfrac{1}{2}u^2 - \tfrac{1}{2}u'^2\right\} = lu\left\{\tfrac{1}{2}u^2 - \tfrac{1}{2}u'^2 - \int_l^{l'} f\,dy\right\}. \qquad \text{......}(24)$$

It should be remarked that it is only for values of y between l and l' that f is effectively involved.

In the special case where $f = \mu y^{-3}$, equations (16), (21) give

$$u^2 l^2 = \mu, \qquad u'^2 l'^2 = \mu, \qquad \text{......................}(25)$$

the introduction of which into (24) shows that, in this case, the loss of energy vanishes; all the conditions can be satisfied, even though there be no dissipation. The reversed motion is then equally admissible.

Experimental.

The formation of bores is illustrated by a very ordinary observation, probably not often thought of in this connection. Something of the kind may usually be seen whenever a stream of water from a tap strikes a horizontal surface [or when water from a can is poured into a flat bath]. The experiment is best made by directing a vertically falling stream into a flat and shallow dish from which the water overflows*. The effective depth may be varied by holding a glass plate in a horizontal position under the water surface. Where the jet strikes, it expands into a thin sheet which diverges for a certain distance, and this distance diminishes as the natural depth of the water over the plate is made greater. The circular boundary where the transition from a small to a greater depth takes place constitutes a bore on a small scale. The flow may be made two-dimensional by limiting it with two battens held in contact with the glass. I have not attempted measures. On the smallest scale surface-tension doubtless plays a considerable part, but this may be minimised by increasing the stream, and correspondingly the depth of the water over the plate, so far as may be convenient.

* The tap that I employed gives a jet whose diameter is 6 mm. A much larger tap may need to be fitted with a special nozzle.—May 14, [1914].

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THE SAND-BLAST.

[*Nature*, Vol. XCIII. p. 188, 1914.]

AMONG the many remarkable anticipations contained in T. Young's *Lectures on Natural Philosophy* (1807) is that in which he explains the effect of what is now commonly known as the sand-blast. On p. 144 he writes:—"There is, however, a limit beyond which the velocity of a body striking another cannot be increased without overcoming its resilience, and breaking it, however small the bulk of the first body may be, and this limit depends on the inertia of the parts of the second body, which must not be disregarded when they are impelled with a considerable velocity. For it is demonstrable that there is a certain velocity, dependent on the nature of a substance, with which the effect of any impulse or pressure is transmitted through it; a certain portion of time, which is shorter accordingly as the body is more elastic, being required for the propagation of the force through any part of it; and if the actual velocity of any impulse be in a greater proportion to this velocity than the extension or compression, of which the substance is capable, is to its whole length, it is obvious that a separation must be produced, since no parts can be extended or compressed which are not yet affected by the impulse, and the length of the portion affected at any instant is not sufficient to allow the required extension or compression. Thus if the velocity with which an impression is transmitted by a certain kind of wood be 15,000 ft. in a second, and it be susceptible of compression to the extent of 1/200 of its length, the greatest velocity that it can resist will be 75 ft. in a second, which is equal to that of a body falling from a height of about 90 ft."

Doubtless this passage was unknown to O. Reynolds when, with customary penetration, in his paper on the sand-blast (*Phil. Mag.* Vol. XLVI. p. 337, 1873) he emphasises that "the intensity of the pressure between bodies on first impact is independent of the size of the bodies."

After his manner, Young was over-concise, and it is not clear precisely what circumstances he had in contemplation. Probably it was the longitudinal impact of bars, and at any rate this affords a convenient example. We may

begin by supposing the bars to be of the same length, material, and section, and before impact to be moving with equal and opposite velocities v. At impact, the impinging faces are reduced to rest, and remain at rest so long as the bars are in contact at all. This condition of rest is propagated in each bar as a wave moving with a velocity a, characteristic of the material. In such a progressive wave there is a general relation between the particle-velocity (estimated relatively to the parts outside the wave) and the compression (e), viz., that the velocity is equal to ae. In the present case the relative particle-velocity is v, so that $v = ae$. The limit of the strength of the material is reached when e has a certain value, and from this the greatest value of v (*half* the original relative velocity) which the bars can bear is immediately inferred.

But the importance of the conclusion depends upon an extension now to be considered. It will be seen that the length of the bars does not enter into the question. Neither does the equality of the lengths. However short one of them may be, we may contemplate an interval after first impact so short that the wave will not have reached the further end, and then the argument remains unaffected. However short one of the impinging bars, the above calculated relative velocity is the highest which the material can bear without undergoing disruption.

As more closely related to practice, the case of two spheres of radii r, r', impinging directly with relative velocity v, is worthy of consideration. According to ordinary elastic theory the only remaining data of the problem are the densities ρ, ρ', and the elasticities. The latter may be taken to be the Young's moduli q, q', and the Poisson's ratios, σ, σ', of which the two last are purely numerical. The same may be said of the ratios q'/q, ρ'/ρ, and r'/r. So far as dimensional quantities are concerned, any maximum strain e may be regarded as a function of r, v, q, and ρ. The two last can occur only in the combination q/ρ, since strain is of no dimensions. Moreover, $q/\rho = a^2$, where a is a velocity. Regarding e as a function of r, v, and a, we see that v and a can occur only as the ratio v/a, and that r cannot appear at all. The maximum strain then is independent of the linear scale; and if the rupture depends only on the maximum strain, it is as likely to occur with small spheres as with large ones. The most interesting case occurs when one sphere is very large relatively to the other, as when a grain of sand impinges upon a glass surface. If the velocity of impact be given, the glass is as likely to be broken by a small grain as by a much larger one. It may be remarked that this conclusion would be upset if rupture depends upon the *duration* of a strain as well as upon its *magnitude*.

The general argument from dynamical similarity that the maximum strain during impact is independent of linear scale, is, of course, not limited to the case of spheres, which has been chosen merely for convenience of statement.

THE EQUILIBRIUM OF REVOLVING LIQUID UNDER CAPILLARY FORCE.

[*Philosophical Magazine*, Vol. XXVIII. pp. 161—170, 1914.]

THE problem of a mass of homogeneous incompressible fluid revolving with uniform angular velocity (ω) and held together by capillary tension (T) is suggested by well-known experiments of Plateau. If there is no rotation, the mass assumes a spherical form. Under the influence of rotation the sphere flattens at the poles, and the oblateness increases with the angular velocity. At higher rotations Plateau's experiments suggest that an annular form may be one of equilibrium. The earlier forms, where the liquid still meets the axis of rotation, have been considered in some detail by Beer*, but little attention seems to have been given to the equilibrium in the form of a ring. A general treatment of this case involves difficulties, but if we assume that the ring is *thin*, viz. that the diameter of the section is small compared with the diameter of the circular axis, we may prove that the form of the section is approximately circular and investigate the small departures from that figure. It is assumed that in the cases considered the surface is one of revolution about the axis of rotation.

Fig. 1 represents a section by a plane through the axis Oy, O being the point where the axis meets the equatorial plane. One of the principal

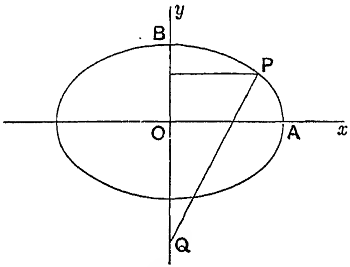

Fig. 1.

* *Pogg. Ann.* Vol. XCVI. p. 210 (1855); compare Poincaré's *Capillarité*, 1895.

curvatures of the surface at P is that of the meridianal curve, the radius of the other principal curvature is PQ the normal as terminated on the axis. The pressure due to the curvature is thus

$$T\left(\frac{1}{\rho}+\frac{1}{PQ}\right),$$

and the equation of equilibrium may be written

$$\frac{1}{\rho}+\frac{1}{PQ}=\frac{\sigma\omega^2x^2}{2T}+\frac{p_0}{T}, \quad \text{.........................(1)}$$

where p_0 is the pressure at points lying upon the axis, and σ is the density of the fluid.

The curvatures may most simply be expressed by means of s, the length of the arc of the curve measured say from A. Thus

$$\frac{1}{PQ}=\frac{1}{x}\frac{dy}{ds}, \quad \frac{1}{\rho}=\frac{d^2y}{dx}\frac{ds^2}{ds}, \quad \text{.......................(2)}$$

so that (1) becomes

$$\frac{dy}{ds}\frac{dx}{ds}+x\frac{d^2y}{ds^2}=\frac{\sigma\omega^2x^3}{2T}\frac{dx}{ds}+\frac{p_0x}{T}\frac{dx}{ds},$$

or on integration

$$x\frac{dy}{ds}=\frac{\sigma\omega^2x^4}{8T}+\frac{p_0x^2}{2T}+\text{const.} \quad \text{.......................(3)}$$

Thus dy/ds is a function of x of known form, say X, and we get for y in terms of x

$$y=\pm\int\frac{Xdx}{\sqrt{(1-X^2)}}, \quad \text{.........................(4)}$$

as given by Beer.

If, as in fig. 1, the curve meets the axis, (3) must be satisfied by $x=0$, $dy/ds=0$. The constant accordingly disappears, and we have the much simplified form

$$\frac{dy}{ds}=\frac{\sigma\omega^2x^3}{8T}+\frac{p_0x}{2T}, \quad \text{.........................(5)}$$

At the point A on the equator $dy/ds=1$. If $OA=a$,

$$1=\frac{\sigma\omega^2a^3}{8T}+\frac{p_0a}{2T};$$

whence eliminating p_0 and writing

$$\Omega=\frac{\sigma\omega^2a^3}{8T}, \quad \text{.........................(6)}$$

we get

$$\frac{dy}{ds}=\Omega\frac{x^3}{a^3}+(1-\Omega)\frac{x}{a}. \quad \text{.........................(7)}$$

In terms of y and x from (7)

$$\pm\frac{dy}{dx} = \frac{x\left(\Omega\frac{x^2}{a^2}+1-\Omega\right)}{\sqrt{\left\{a^2 - x^2\left(\Omega\frac{x^2}{a^2}+1-\Omega\right)^2\right\}}}, \quad \text{.................(8)}$$

or if we write

$$x^2/a^2 = 1 - z, \quad \text{..............................(9)}$$

$$-\frac{2\,dy}{a\,dz} = \frac{1-\Omega z}{\sqrt{z}\,.\sqrt{\{1+2(1-z)\,\Omega - z(1-z)\,\Omega^2\}}}$$

$$= (1-\Omega+\tfrac{3}{2}\Omega^2)\,z^{-\frac{1}{2}} - \tfrac{3}{2}\Omega^2 z^{\frac{1}{2}}, \quad \text{....................(10)}$$

when we neglect higher powers of Ω than Ω^2. Reverting to x, we find for the integral of (10)

$$\pm\frac{y}{a} = (1-\Omega+\tfrac{3}{2}\Omega^2)\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{\Omega^2}{2}\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}, \quad \text{...........(11)}$$

no constant being added since $y = 0$ when $x = a$.

If we stop at Ω, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-\Omega)^2} = 1 \quad \text{..........................(12)}$$

representing an ellipse whose minor axis OB is $a(1-\Omega)$.

When Ω^2 is retained,

$$OB = (1-\Omega+\Omega^2)\,a. \quad \text{......................(13)}$$

The approximation in powers of Ω could of course be continued if desired.

So long as $\Omega < 1$, p_0 is positive and the (equal) curvatures at B are convex. When $\Omega = 1$, $p_0 = 0$ and the surface at B is flat. In this case (8) gives

$$\pm\frac{dy}{dx} = \frac{x^3}{\sqrt{\{a^6 - x^6\}}}, \quad \text{..........................(14)}$$

or if we set $x = a\sin^{\frac{1}{3}}\phi$,

$$\pm\frac{dy}{d\phi} = \frac{a}{3}\sin^{\frac{1}{3}}\phi. \quad \text{..........................(15)}$$

Here $x = a$ corresponds to $\phi = \frac{1}{2}\pi$, and $x = 0$ corresponds to $\phi = 0$. Hence

$$OB = \frac{a}{3}\int_0^{\frac{1}{2}\pi}\sin^{\frac{1}{3}}\phi\,d\phi. \quad \text{......................(16)}$$

The integral in (16) may be expressed in terms of gamma functions and we get

$$OB = a\sqrt{\pi}\,.\,\Gamma(\tfrac{2}{3}) \div \Gamma(\tfrac{1}{6}) = \cdot 4312\,a. \quad \text{................(17)}$$

When $\Omega > 1$, the curvature at B is concave and p_0 is negative, as is quite permissible.

In order to trace the various curves we may calculate by quadratures from (4) the position of a sufficient number of points. This, as I understand, was the procedure adopted by Beer. An alternative method is to trace the curves by direct use of the radius of curvature at the point arrived at. Starting from (7) we find

$$\frac{d^2y}{ds^2} = \left(\Omega\frac{3x^2}{a^3} + \frac{1-\Omega}{a}\right)\frac{dx}{ds},$$

and thence

$$\frac{a}{\rho} = a\,\frac{d^2y/ds^2}{dx/ds} = \Omega\frac{3x^2}{a^2} + 1 - \Omega. \quad\text{......................(18)}$$

From (18) we see at once that $\Omega = 0$ makes $\rho = a$ throughout, and that when $\Omega = 1$, $x = 0$ makes $\rho = \infty$.

In tracing a curve we start from the point A in a known direction and with $\rho = a/(2\Omega + 1)$, and at every point arrived at we know with what curvature to proceed. If, as has been assumed, the curve meets the axis, it must do so at right angles, and a solution is then obtained.

The method is readily applied to the case $\Omega = 1$ with the advantage that we know where the curve should meet the axis of y. From (18) with $\Omega = 1$ and $a = 5$,

$$\frac{1}{\rho} = \frac{24x^2}{1000}. \quad\text{..............................(19)}$$

Starting from $x = 5$ we draw small portions of the curve corresponding to decrements of x equal to ·2, thus arriving in succession at the points for which $x = 4{\cdot}8$, $4{\cdot}6$, $4{\cdot}4$, &c. For these portions we employ the *mean* curvatures, corresponding to $x = 4{\cdot}9$, $4{\cdot}7$, &c. calculated from (19). It is convenient to use squared paper and fair results may be obtained with the ordinary ruler and compasses. There is no need actually to draw the normals. But for such work the procedure recommended by Boys* offers great advantages. The ruler and compasses are replaced by a straight scale divided upon a strip of semi-transparent celluloid. At one point on the scale a fine pencil point protrudes through a small hole and describes the diminutive circular arc. Another point of the scale at the required distance occupies the centre of the circle and is held temporarily at rest with the aid of a small brass tripod standing on sharp needle points. After each step the celluloid is held firmly to the paper and the tripod is moved to the point of the scale required to give the next value of the curvature. The ordinates of the curve so drawn are given in the second and fifth columns of the annexed table. It will be seen that from $x = 0$ to $x = 2$ the curve is very flat. Fig. (1).

* *Phil. Mag.* Vol. XXXVI. p. 75 (1893). I am much indebted to Mr Boys for the loan of suitable instruments. The use is easy after a little practice.

Another case of special interest is the last figure reaching the axis of symmetry at all, which occurs at the point $x = 0$. We do not know beforehand to what value of Ω this corresponds, and curves must be drawn tentatively. It appears that $\Omega = 2{\cdot}4$ approximately, and the values of y obtained from this curve are given in columns 3 and 6 of the table. Fig. (2)*.

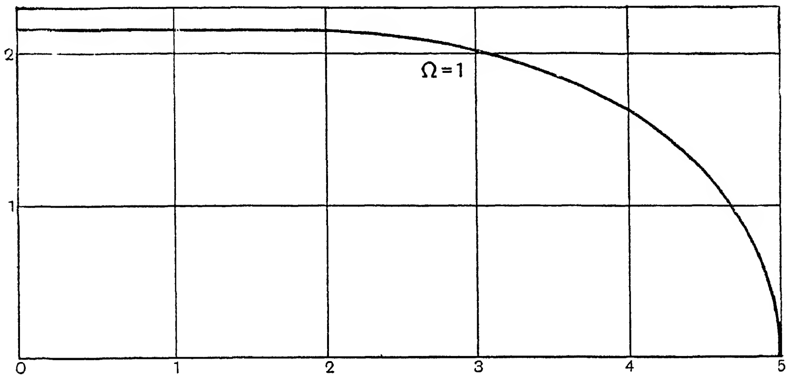

Fig. (1).

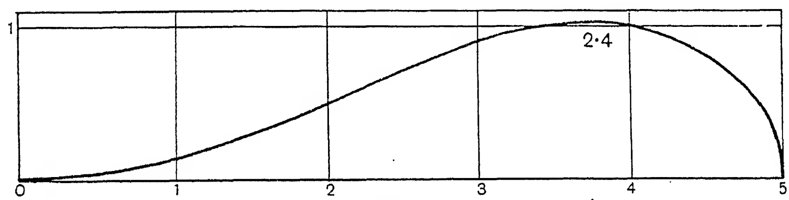

Fig. (2).

$\pm x$	$\pm y$	$\pm y'$	$\pm x$	$\pm y$	$\pm y'$
0·0	2·16	0·00	2·6	2·06	0·75
0·2	2·16	0·01	2·8	2·03	0·83
0·4	2·16	0·03	3·0	1·99	0·90
0·6	2·16	0·06	3·2	1·95	0·95
0·8	2·16	0·10	3·4	1·89	0·99
1·0	2·15	0·14	3·6	1·81	1·01
1·2	2·15	0·20	3·8	1·72	1·02
1·4	2·15	0·27	4·0	1·61	1·00
1·6	2·15	0·34	4·2	1·49	0·98
1·8	2·14	0·42	4·4	1·32	0·89
2·0	2·12	0·50	4·6	1·11	0·78
2·2	2·11	0·58	4·8	0·80	0·67
2·4	2·09	0·65	4·9	0·59	0·41
			5·0	0·00	0·00

There is a little difficulty in drawing the curve through the point of zero curvature. I found it best to begin at both ends $(x = 0, y = 0)$ and $(x = 5, y = 0)$ with an assumed value of Ω and examine whether the two parts could be made to fit.

* [1916. These figures were omitted in the original memoir.]

When $\Omega > 2{\cdot}4$ and the curve does not meet the axis at all, the constant in (3) must be retained, and the difficulty is much increased. If we suppose that $dy/ds = +1$ when $x = a_2$ and $dy/ds = -1$ when $x = a_1$, we can determine p_0 as well as the constant of integration, and (3) becomes

$$x\frac{dy}{ds} = \frac{\sigma\omega^2}{8T}(x^2 - a_1^2)(x^2 - a_2^2) + \frac{x^2 - a_1 a_2}{a_2 - a_1}. \quad \text{...........(20)}$$

We may imagine a curve to be traced by means of this equation. We start from the point A where $y = 0$, $x = a_2$ and in the direction perpendicular to OA, and (as before) we are told in what direction to proceed at any point reached. When $x = a_1$, the tangent must again be parallel to the axis, but there is nothing to ensure that this occurs when $y = 0$. To secure this end and so obtain an annular form of equilibrium, $\sigma\omega^2/T$ must be chosen suitably, but there is no means apparent of doing this beforehand. The process of curve tracing can only be tentative.

If we form the expression for the curvature as before, we obtain

$$\frac{1}{\rho} = \frac{\sigma\omega^2}{8T}\left(3x^2 - a_1^2 - a_2^2 - \frac{a_1^2 a_2^2}{x^2}\right) + \frac{1}{a_2 - a_1} + \frac{a_1 a_2}{x^2(a_2 - a_1)} \quad \text{......(21)}$$

by means of which the curves may be traced tentatively.

If we retain the normal PQ, as we may conveniently do in using Boys' method, we have the simpler expression

$$\frac{1}{\rho} + \frac{1}{PQ} = \frac{\sigma\omega^2}{4T}(2x^2 - a_1^2 - a_2^2) + \frac{2}{a_2 - a_1}. \quad \text{...........(22)}$$

When the radius CP of the section is very small in comparison with the radius of the ring OC, the conditions are approximately satisfied by a circular

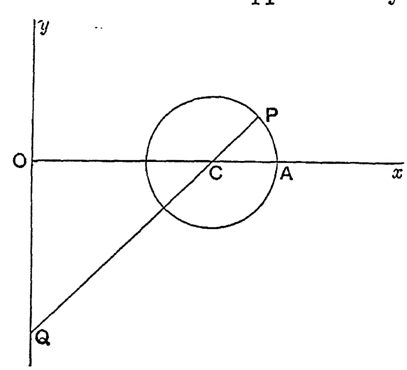

Fig. 2.

form. We write $CP = r$, $OC = a$, $PCA = \theta$. Then, r being supposed constant, the principal radii of curvature are r and $a\sec\theta + r$, so that the equation of equilibrium is

$$\frac{p_0}{T} = \frac{1}{r} + \frac{\cos\theta}{a + r\cos\theta} - \frac{\omega^2}{2T}(a + r\cos\theta)^2, \quad \text{..............(23)}$$

in which p_0 should be constant as θ varies. In this

$$\frac{\cos\theta}{a + r\cos\theta} = \frac{1}{a}\left\{-\frac{r}{2a} + \left(1 + \frac{3r^2}{4a^2}\right)\cos\theta - \frac{r}{2a}\cos 2\theta + \frac{r^2}{4a^2}\cos 3\theta\right\},$$

$$\left(1 + \frac{r}{a}\cos\theta\right)^2 = 1 + \frac{r^2}{2a^2} + \frac{2r}{a}\cos\theta + \frac{r^2}{2a^2}\cos 2\theta.$$

Thus approximately

$$\frac{ap_0}{T} = \frac{a}{r} - \frac{r}{2a} - \frac{\omega^2 a^3}{2T}\left(1 + \frac{r^2}{2a^2}\right) + \cos\theta\left\{1 + \frac{3r^2}{4a^2} - \frac{\omega^2 a^3}{2T}\frac{2r}{a}\right\}$$

$$+ \cos 2\theta\left\{-\frac{r}{2a} - \frac{\omega^2 a}{2T}\frac{r^2}{2a^2}\right\} + \cos 3\theta \, . \, \frac{r^2}{4a^2}. \quad \text{...(24)}$$

The term in $\cos\theta$ will vanish if we take ω so that

$$\frac{\omega^2 a^2}{T} = \frac{1}{r}\left(1 - \frac{3r^2}{a^2}\right). \quad \text{......................(25)}$$

The coefficient of $\cos 2\theta$ then becomes

$$-\frac{3r}{4a} + \text{cubes of } \frac{r}{a}. \quad \text{......................(26)}$$

If we are content to neglect r/a in comparison with unity, the condition of equilibrium is satisfied by the circular form; otherwise there is an inequality of pressure of this order in the term proportional to $\cos 2\theta$. From (25) it is seen that if a and T be given, the necessary angular velocity increases as the radius of the section decreases.

In order to secure a better fulfilment of the pressure equation it is necessary to suppose r variable, and this of course complicates the expressions for the curvatures. For that in the meridianal plane we have

$$\frac{1}{\rho} = \frac{r^2 - r\dfrac{d^2r}{d\theta^2} + 2\left(\dfrac{dr}{d\theta}\right)^2}{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}},$$

or with sufficient approximation

$$\frac{1}{\rho} = \frac{1}{r}\left\{1 - \frac{1}{r}\frac{d^2r}{d\theta^2} + \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\}. \quad \text{................(27)}$$

For the curvature in the perpendicular plane we have to substitute PQ', measured along the normal, for PQ, whose expression remains as before (fig. 3). Now

$$\frac{PQ}{PQ'} = \frac{\sin Q'}{\sin Q} = \cos QPQ' - \tan\theta \sin QPQ'$$

in which

$$\cos QPQ' = \frac{CN}{r} = \left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}^{-\frac{1}{2}} = 1 - \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2$$

approximately,

$$\sin QPQ' = -\frac{1}{r}\frac{dr}{d\theta}\left\{1 - \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\}.$$

Thus

$$\frac{1}{PQ'} = \frac{\cos\theta}{a + r\cos\theta}\left\{1 - \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\} + \frac{\sin\theta}{a + r\cos\theta}\frac{1}{r}\frac{dr}{d\theta}\left\{1 - \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\} \text{ [illegible]}$$

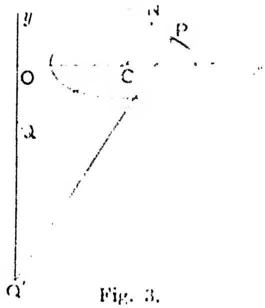

Fig. 3.

It will be found that it is unnecessary to retain $(dr/d\theta)^2$, and the pressure equation becomes

$$\frac{ap_0}{T} = \frac{a}{r}\left\{1 - \frac{1}{r}\frac{d^2r}{d\theta^2}\right\} + \frac{a\cos\theta}{a + r\cos\theta} + \frac{a\sin\theta}{a + r\cos\theta}\frac{1}{r}\frac{dr}{d\theta} - \frac{\omega^2 a}{2T}\left(1 + \frac{r\cos\theta}{a}\right)^2 \text{ [illegible]} \quad (2)$$

It is proposed to satisfy this equation so far as terms of the order [illegible] inclusive.

As a function of θ, r may be taken to be

$$r = r_0 + \delta r = r_0 + r_1\cos\theta + r_2\cos 2\theta + \ldots \qquad \ldots (3)$$

where r_1, r_2, &c. are constants small relatively to r_0. It will appear that in our order of approximation $(\delta r/r_0)^2$ may be neglected and that it is unnecessary to include the r's beyond r_3 inclusive. We have

$$\frac{a}{r} = \frac{a}{r_0}\left\{1 - \frac{\delta r}{r_0} + \left(\frac{\delta r}{r_0}\right)^2\right\},$$

$$\frac{a\cos\theta}{a + r\cos\theta} = -\frac{r_0}{2a} - \frac{r_2}{4a} + \cos\theta\left\{1 + \frac{3r_0^2}{4a^2} - \frac{3r_1}{4a} - \frac{r_3}{4a}\right\}$$

$$- \cos 2\theta\left\{\frac{r_0}{2a} + \frac{r_2}{2a}\right\} + \cos 3\theta \text{ [illegible]}$$

$$\left(1 + \frac{r\cos\theta}{a}\right)^2 = 1 + \frac{r_1}{a^2} + \frac{r_0^2}{2a^2} + \frac{r_0 r_2}{2a^2} + \cos\theta \text{ [illegible]}$$

$$+ \cos 2\theta\left\{\frac{r_1}{a} + \frac{r_3}{a} + \frac{r_0^2}{2a^2} + \frac{r_0 r_2}{a^2}\right\} + \cos 3\theta \text{ [illegible]}$$

$$-\frac{a}{r^2}\frac{d^2r}{d\theta^2}=\frac{a}{r_0^2}\left\{r_1\cos\theta+4r_2\cos 2\theta+9r_3\cos 3\theta\right\},$$

$$\frac{a\sin\theta}{a+r\cos\theta}\frac{1}{r}\frac{dr}{d\theta}=-\frac{r_1}{2r_0}+\frac{r_2}{2a}+\cos\theta\left\{\frac{r_1}{4a}-\frac{r_2}{r_0}+\frac{3r_3}{4a}\right\}$$

$$+\cos 2\theta\left\{\frac{r_1}{2r_0}-\frac{3r_3}{2r_0}\right\}+\cos 3\theta\left\{\frac{r_2}{r_0}-\frac{r_1}{4a}\right\}.$$

Thus altogether for the coefficient of $\cos\theta$ on the right of (29) we get

$$1+\frac{3r_0^2}{4a^2}-\frac{r_1}{2a}-\frac{r_2}{r_0}-\frac{\omega^2a^3}{2T}\left\{\frac{2r_0}{a}+\frac{r_2}{a}\right\}.$$

This will be made to vanish if we take ω such that

$$\frac{\omega^2a^2r_0}{T}=1+\frac{3r_0^2}{4a^2}-\frac{r_1}{2a}-\frac{3r_2}{2r_0}. \quad\text{.....................(31)}$$

The coefficient of $\cos 2\theta$ is

$$\frac{3ar_2}{r_0^2}-\frac{r_0}{2a}+\frac{r_1}{2r_0}-\frac{3r_3}{2r_0}-\frac{\omega^2a^3}{2T}\left\{\frac{r_1}{a}+\frac{r_3}{a}+\frac{r_0^2}{2a^2}\right\},$$

or when we introduce the value of ω from (31)

$$\frac{3ar_2}{r_0^2}-\frac{3r_0}{4a}-\frac{2r_3}{r_0}. \quad\text{..........................(32)}$$

The coefficient of $\cos 3\theta$ is in like manner

$$\frac{8ar_3}{r_0^2}+\frac{r_0^2}{4a^2}+\frac{r_2}{2r_0}. \quad\text{..........................(33)}$$

These coefficients are annulled and ap_0/T is rendered constant so far as the second order of r_0/a inclusive, when we take r_4, r_5, &c. equal to zero and

$$r_2/r_0=r_0^2/4a^2, \qquad r_3/r_0=-3r_0^3/64a^3. \quad\text{..............(34)}$$

We may also suppose that $r_1=0$.

The solution of the problem is accordingly that

$$r=r_0\left\{1+\frac{r_0^2}{4a^2}\cos 2\theta-\frac{3r_0^3}{64a^3}\cos 3\theta\right\} \quad\text{..............(35)}$$

gives the figure of equilibrium, provided ω be such that

$$\frac{\omega^2a^2r_0}{T}=1+\frac{3r_0^2}{8a^2}. \quad\text{..........................(36)}$$

The form of a thin ring of equilibrium is thus determined; but it seems probable that the equilibrium would be unstable for disturbances involving a departure from symmetry round the axis of revolution.

approximately,

$$\sin QPQ' = -\frac{1}{r}\frac{dr}{d\theta}\left\{1 - \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\}.$$

Thus

$$\frac{1}{PQ'} = \frac{\cos\theta}{a + r\cos\theta}\left\{1 - \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\} + \frac{\sin\theta}{a + r\cos\theta}\,\frac{1}{r}\frac{dr}{d\theta}\left\{1 - \frac{1}{2r^2}\left(\frac{dr}{d\theta}\right)^2\right\}. \qquad \text{.........(28)}$$

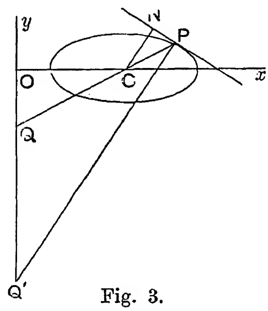

Fig. 3.

It will be found that it is unnecessary to retain $(dr/d\theta)^2$, and thus the pressure equation becomes

$$\frac{ap_0}{T} = \frac{a}{r}\left\{1 - \frac{1}{r}\frac{d^2r}{d\theta^2}\right\} + \frac{a\cos\theta}{a + r\cos\theta} + \frac{a\sin\theta}{a + r\cos\theta}\,\frac{1}{r}\frac{dr}{d\theta} - \frac{\omega^2a^3}{2T}\left(1 + \frac{r\cos\theta}{a}\right)^2. \qquad \text{.........(29)}$$

It is proposed to satisfy this equation so far as terms of the order r^2/a^2 inclusive.

As a function of θ, r may be taken to be

$$r = r_0 + \delta r = r_0 + r_1\cos\theta + r_2\cos 2\theta + \ldots, \qquad \text{...........(30)}$$

where r_1, r_2, &c. are constants small relatively to r_0. It will appear that to our order of approximation $(\delta r/r_0)^2$ may be neglected and that it is unnecessary to include the r's beyond r_3 inclusive. We have

$$\frac{a}{r} = \frac{a}{r_0}\left\{1 - \frac{\delta r}{r_0} + \left(\frac{\delta r}{r_0}\right)^2\right\},$$

$$\frac{a\cos\theta}{a + r\cos\theta} = -\frac{r_0}{2a} - \frac{r_2}{4a} + \cos\theta\left\{1 + \frac{3r_0^2}{4a^2} - \frac{3r_1}{4a} - \frac{r_3}{4a}\right\}$$
$$- \cos 2\theta\left\{\frac{r_0}{2a} + \frac{r_2}{2a}\right\} + \cos 3\theta\left\{\frac{r_0^2}{4a^2} - \frac{r_1}{4a} - \frac{r_3}{2a}\right\},$$

$$\left(1 + \frac{r\cos\theta}{a}\right)^2 = 1 + \frac{r_1}{a^2} + \frac{r_0^2}{2a^2} + \frac{r_0r_2}{2a^2} + \cos\theta\left\{\frac{2r_0}{a} + \frac{r_2}{a} + \frac{3r_0r_1}{2a^2} + \frac{r_0r_3}{2a^2}\right\}$$
$$+ \cos 2\theta\left\{\frac{r_1}{a} + \frac{r_3}{a} + \frac{r_0^2}{2a^2} + \frac{r_0r_2}{a^2}\right\} + \cos 3\theta\left\{\frac{r_2}{a} + \frac{r_0r_3}{a^2} + \frac{r_0r_1}{2a^2}\right\},$$

$$-\frac{a}{r^2}\frac{d^2r}{d\theta^2} = \frac{a}{r_0^2}\left\{r_1 \cos\theta + 4r_2\cos 2\theta + 9r_3\cos 3\theta\right\},$$

$$\frac{a\sin\theta}{a + r\cos\theta}\frac{1}{r}\frac{dr}{d\theta} = -\frac{r_1}{2r_0} + \frac{r_2}{2a} + \cos\theta\left\{\frac{r_1}{4a} - \frac{r_2}{r_0} + \frac{3r_3}{4a}\right\}$$

$$+ \cos 2\theta\left\{\frac{r_1}{2r_0} - \frac{3r_3}{2r_0}\right\} + \cos 3\theta\left\{\frac{r_2}{r_0} - \frac{r_1}{4a}\right\}.$$

Thus altogether for the coefficient of $\cos\theta$ on the right of (29) we get

$$1 + \frac{3r_0^2}{4a^2} - \frac{r_1}{2a} - \frac{r_2}{r_0} - \frac{\omega^2 a^3}{2T}\left\{\frac{2r_0}{a} + \frac{r_2}{a}\right\}.$$

This will be made to vanish if we take ω such that

$$\frac{\omega^2 a^2 r_0}{T} = 1 + \frac{3r_0^2}{4a^2} - \frac{r_1}{2a} - \frac{3r_2}{2r_0}. \quad \text{.....................(31)}$$

The coefficient of $\cos 2\theta$ is

$$\frac{3ar_2}{r_0^2} - \frac{r_0}{2a} + \frac{r_1}{2r_0} - \frac{3r_3}{2r_0} - \frac{\omega^2 a^3}{2T}\left\{\frac{r_1}{a} + \frac{r_3}{a} + \frac{r_0^2}{2a^2}\right\},$$

or when we introduce the value of ω from (31)

$$\frac{3ar_2}{r_0^2} - \frac{3r_0}{4a} - \frac{2r_3}{r_0}. \quad \text{..........................(32)}$$

The coefficient of $\cos 3\theta$ is in like manner

$$\frac{8ar_3}{r_0^2} + \frac{r_0^2}{4a^2} + \frac{r_2}{2r_0}. \quad \text{..........................(33)}$$

These coefficients are annulled and ap_0/T is rendered constant so far as the second order of r_0/a inclusive, when we take r_4, r_5, &c. equal to zero and

$$r_2/r_0 = r_0^2/4a^2, \qquad r_3/r_0 = -3r_0^3/64a^3. \quad \text{..............(34)}$$

We may also suppose that $r_1 = 0$.

The solution of the problem is accordingly that

$$r = r_0\left\{1 + \frac{r_0^2}{4a^2}\cos 2\theta - \frac{3r_0^3}{64a^3}\cos 3\theta\right\} \quad \text{..............(35)}$$

gives the figure of equilibrium, provided ω be such that

$$\frac{\omega^2 a^2 r_0}{T} = 1 + \frac{3r_0^2}{8a^2}. \quad \text{..........................(36)}$$

The form of a thin ring of equilibrium is thus determined; but it seems probable that the equilibrium would be unstable for disturbances involving a departure from symmetry round the axis of revolution.

FURTHER REMARKS ON THE STABILITY OF VISCOUS FLUID MOTION.

[*Philosophical Magazine*, Vol. XXVIII. pp. 609—619, 1914.]

AT an early date my attention was called to the problem of the stability of fluid motion in connexion with the acoustical phenomena of sensitive jets, which may be ignited or unignited. In the former case they are usually referred to as sensitive *flames*. These are naturally the more conspicuous experimentally, but the theoretical conditions are simpler when the jets are unignited, or at any rate not ignited until the question of stability has been decided.

The instability of a surface of separation in a non-viscous liquid, *i.e.* of a surface where the velocity is discontinuous, had already been remarked by Helmholtz, and in 1879 I applied a method, due to Kelvin, to investigate the character of the instability more precisely. But nothing very practical can be arrived at so long as the original steady motion is treated as discontinuous, for in consequence of viscosity such a discontinuity in a real fluid must instantly disappear. A nearer approach to actuality is to suppose that while the *velocity* in a laminated steady motion is continuous, the *rotation* or vorticity changes suddenly in passing from one layer of finite thickness to another. Several problems of this sort have been treated in various papers*. The most general conclusion may be thus stated. The steady motion of a non-viscous liquid in two dimensions between fixed parallel plane walls is stable provided that the velocity U, everywhere parallel to the walls and a function of y only, is such that d^2U/dy^2 is of one sign throughout, y being the coordinate measured perpendicularly to the walls. It is here assumed that the disturbance is in two dimensions and *infinitesimal*. It involves

* *Proc. Lond. Math. Soc.* Vol. X. p. 4 (1879); XI. p. 57 (1880); XIX. p. 67 (1887); XXVII. p. 5 (1895); *Phil. Mag.* Vol. XXXIV. p. 59 (1892); XXVI. p. 1001 (1913); *Scientific Papers*, Arts. 58, 66, 144, 216, 194. [See also Art. 377.]

a slipping at the walls, but this presents no inconsistency so long as the fluid is regarded as absolutely non-viscous.

The steady motions for which stability in a non-viscous fluid may be inferred include those assumed by a viscous fluid in two important cases, (i) the simple shearing motion between two planes for which $d^2U/dy^2 = 0$, and (ii) the flow (under suitable forces) between two fixed plane walls for which d^2U/dy^2 is a finite constant. And the question presented itself whether the effect of viscosity upon the disturbance could be to introduce instability. An affirmative answer, though suggested by common experience and the special investigations of O. Reynolds*, seemed difficult to reconcile with the undoubted fact that great viscosity makes for stability.

It was under these circumstances that "the Criterion of the Stability and Instability of the Motion of a Viscous Fluid," with special reference to cases (i) and (ii) above, was proposed as the subject of an Adams Prize essay†, and shortly afterwards the matter was taken up by Kelvin‡ in papers which form the foundation of much that has since been written upon the subject. His conclusion was that in both cases the steady motion is wholly stable for infinitesimal disturbances, whatever may be the value of the viscosity (μ); but that when the disturbances are finite, the limits of stability become narrower and narrower as μ diminishes. Two methods are employed: the first a special method applicable only to case (i) of a simple shear, the second (ii) more general and applicable to both cases. In 1892 (*l.c.*) I had occasion to take exception to the proof of stability by the second method, and Orr§ has since shown that the same objection applies to the special method. Accordingly Kelvin's proof of stability cannot be considered sufficient, even in case (i). That Kelvin himself (partially) recognized this is shown by the following interesting and characteristic letter, which I venture to give in full.

July 10 (? 1895).

"On Saturday I saw a splendid illustration by Arnulf Mallock of our ideas regarding instability of water between two parallel planes, one kept moving and the other fixed. (Fig. 1) Coaxal cylinders, nearly enough planes for our illustration. The rotation of the outer can was kept very accurately uniform at whatever speed the governor was set for, when left to itself. At one of the speeds he shewed me, the water came to regular regime, *quite smooth*. I dipped a disturbing rod an inch or two down into the water and immediately the torque increased largely. *Smooth* regime could only be

* *Phil. Trans.* 1883, Part III. p. 935.

† *Phil. Mag.* Vol. XXIV. p. 142 (1887). The suggestion came from me, but the notice was (I think) drawn up by Stokes.

‡ *Phil. Mag.* Vol. XXIV. pp. 188, 272 (1887); *Collected Papers*, Vol. IV. p. 321.

§ Orr, *Proc. Roy. Irish Acad.* Vol. XXVII. (1907).

re-established by slowing down and bringing up to speed again, gradually enough.

"Without the disturbing rod at all, I found that by resisting the outer can by hand somewhat suddenly, but not very much so, the torque increased suddenly and the motion became visibly turbulent at the lower speed and remained so.

"I have no doubt we should find with higher and higher speeds, very gradually reached, stability of laminar or non-turbulent motion, but with narrower and narrower limits as to magnitude of disturbance; and so find through a large range of velocity, a confirmation of *Phil. Mag.* 1887, 2, pp. 191—196. The experiment would, at high velocities, fail to prove the stability which the mathematical investigation proves for every velocity however high.

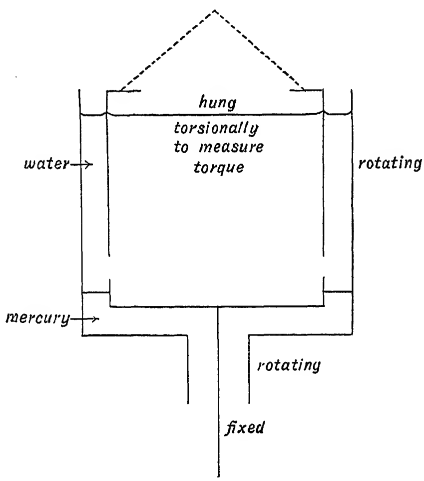

Fig. 1.

"As to *Phil. Mag.* 1887, 2, pp. 272—278, I admit that the mathematical proof is not complete, and withdraw [temporarily ?] the words 'virtually inclusive' (p. 273, line 3). I still think it probable that the laminar motion is stable for this case also. In your (*Phil. Mag.* July 1892, pp. 67, 68) refusal to admit that stability is proved you don't distinguish the case in which my proof was complete from the case in which it seems, and therefore is, not complete.

"Your equation (24) of p. 68 is only valid for infinitely small motion, in which the squares of the total velocities are everywhere negligible; and in this case the motion is manifestly periodic, for any stated periodic conditions of the boundary, and comes to rest according to the logarithmic law if the boundary is brought to rest at any time.

"In your p. 62, lines 11 and 12 are 'inaccurate.' Stokes limits his investigation to the case in which the squares of the velocities can be neglected

$$\left(i.e. \frac{\text{radius of globe} \times \text{velocity}}{\text{diffusivity}} \text{ very small}\right),$$

in which it is manifest that the steady motion is the same whatever the viscosity; but it is manifest that when the squares cannot be neglected, the steady motion is very different (and horribly difficult to find) for different degrees of viscosity.

"In your p. 62, near the foot, it is not explained what V is; and it disappears henceforth.—Great want of explanation here—Did you not want your paper to be understandable without Basset in hand? I find your two papers of July/92, pp. 61—70, and Oct./93, pp. 355—372, very difficult reading, in every page, and in some ∞ ly difficult.

"Pp. 366, 367 very mysterious. The elastic problem is not defined. It is impossible that there can be the rectilineal motion of the fluid asserted in p. 367, lines 17—19 from foot, in circumstances of motion, quite undefined, but of some kind making the lines of motion on the right side different from those on the left. The conditions are not explained for either the elastic-solid*, or the hydraulic case.

"See p. 361, lines 19, 20, 21 from foot. The formation of a backwater depends essentially on the non-negligibility of squares of velocities; and your p. 367, lines 1—4, and line 17 from foot, are not right.

"If you come to the R. S. Library Committee on Thursday we may come to agreement on some of these questions."

Although the main purpose in Kelvin's papers of 1887 was not attained, his special solution for a disturbed vorticity in case (i) is not without interest. The general dynamical equation for the vorticity in two dimensions is

$$\frac{D\zeta}{Dt} = \frac{d\zeta}{dt} + u\frac{d\zeta}{dx} + v\frac{d\zeta}{dy} = \nu\nabla^2\zeta, \quad \text{.....................(1)}$$

where $\nu\,(=\mu/\rho)$ is the kinematic viscosity and $\nabla^2 = d^2/dx^2 + d^2/dy^2$. In this hydrodynamical equation ζ is itself a feature of the motion, being connected with the velocities u, v by the relation

$$\zeta = \tfrac{1}{2}\left(\frac{du}{dy} - \frac{dv}{dx}\right), \quad \text{..............................(2)}$$

while u, v themselves satisfy the "equation of continuity"

$$\frac{du}{dx} + \frac{dv}{dy} = 0. \quad \text{..............................(3)}$$

* I think Kelvin did not understand that the analogous elastic problem referred to is that of a thin *plate*. See words following equation (5) of my paper.

In other applications of (1), *e.g.* to the diffusion of heat or dissolved matter in a moving fluid, ζ is a new dependent variable, not subject to (2), and representing temperature or salinity. We may then regard the motion as known while ζ remains to be determined. In any case $\frac{1}{2} D\zeta^2/Dt = \nu \zeta \nabla^2 \zeta$. If the fluid move within fixed boundaries, or extend to infinity under suitable conditions, and we integrate over the area included,

$$\iint \frac{D\zeta^2}{Dt} dx\, dy = \frac{d}{dt} \iint \zeta^2 \, dx\, dy,$$

so that

$$\frac{1}{2} \frac{d}{dt} \iint \zeta^2 \, dx\, dy = \nu \iint \zeta \nabla^2 \zeta \, dx\, dy = \nu \int \zeta \frac{d\zeta}{dn} ds - \nu \iint \left\{ \left(\frac{d\zeta}{dx}\right)^2 + \left(\frac{d\zeta}{dy}\right)^2 \right\} dx\, dy, \quad \text{.........(4)}$$

by Green's theorem. The boundary integral disappears, if either ζ or $d\zeta/dn$ there vanishes, and then the integral on the left necessarily diminishes as time progresses*. The same conclusion follows if ζ and $d\zeta/dn$ have all along the boundary contrary signs. Under these conditions ζ tends to zero over the whole of the area concerned. The case where at the boundary ζ is required to have a constant finite value Z is virtually included, since if we write $Z + \zeta'$ for ζ, Z disappears from (1), and ζ everywhere tends to the value Z.

In the hydrodynamical problem of the simple shearing motion, ζ is a constant, say Z, u is a linear function of y, say U, and $v = 0$. If in the disturbed motion the vorticity be $Z + \zeta$, and the components of velocity be $U + u$ and v, equation (1) becomes

$$\frac{d\zeta}{dt} + (U + u)\frac{d\zeta}{dx} + v\frac{d\zeta}{dy} = \nu \nabla^2 \zeta, \quad \text{....................(5)}$$

in which ζ, u, and v relate to the *disturbance*. If the disturbance be treated as infinitesimal, the terms of the second order are to be omitted and we get simply

$$\frac{d\zeta}{dt} + U\frac{d\zeta}{dx} = \nu \nabla^2 \zeta. \quad \text{............................(6)}$$

In (6) the motion of the fluid, represented by U simply, is given independently of ζ, and the equation is the same as would apply if ζ denoted the temperature, or salinity, of the fluid moving with velocity U. Any conclusions that we may draw have thus a widened interest.

In Kelvin's solution of (6) the disturbance is supposed to be periodic in x, proportional to e^{ikx}, and U is taken equal to βy. He assumes for trial

$$\zeta = T e^{i\{kx + (n - k\beta t) y\}}, \quad \text{............................(7)}$$

* Compare Orr, *l.c.* p. 115.

where T is a function of t. On substitution in (6) he finds

$$\frac{dT}{dt} = -\nu \{k^2 + (n - k\beta t)^2\} T,$$

whence
$$T = Ce^{-\nu t\{k^2+n^2-nk\beta t+\frac{1}{3}k^2\beta^2t^2\}}, \quad \text{......................(8)}$$

and comes ultimately to zero. Equations (7) and (8) determine ζ and so suffice for the heat and salinity problems in an infinitely extended fluid. As an example, if we suppose $n = 0$ and take the real part of (7),

$$\zeta = T \cos k(x - \beta t \,.\, y), \quad \text{........................(9)}$$

reducing to $\zeta = C \cos kx$ simply when $t = 0$. At this stage the lines of constant ζ are parallel to y. As time advances, T diminishes with increasing rapidity, and the lines of constant ζ tend to become parallel to x. If x be constant, ζ varies more and more rapidly with y. This solution gives a good idea of the course of events when a liquid of unequal salinity is stirred.

In the hydrodynamical problem we have further to deduce the small velocities u, v corresponding to ζ. From (2) and (3), if u and v are proportional to e^{ikx},

$$\zeta = \frac{i}{2k}\left(\frac{d^2v}{dy^2} - k^2v\right) = \frac{i}{2k}\nabla^2 v. \quad \text{....................(10)}$$

Thus, corresponding to (9),

$$v = -\frac{2T}{k(1+\beta^2t^2)} \sin k(x - \beta t \,.\, y). \quad \text{................(11)}$$

No complementary terms satisfying $d^2v/dy^2 - k^2v = 0$ are admissible, on account of the assumed periodicity with x. It should be mentioned that in Kelvin's treatment the disturbance is not limited to be two-dimensional.

Another remarkable solution for an unlimited fluid of Kelvin's equation (6) with $U = \beta y$ has been given by Oseen*. In this case the initial value of ζ is concentrated at one point (ξ, η), and the problem may naturally be regarded as an extension of one of Fourier relating to the conduction of heat. Oseen finds

$$\zeta(x, y, t) = \frac{Ce^{-\frac{\{\xi - x + \frac{1}{2}\beta t(\eta + y)\}^2}{4\nu t(1+\frac{1}{12}\beta^2t^2)} - \frac{(\eta - y)^2}{4\nu t}}}{4\pi\nu t\sqrt{(1+\frac{1}{12}\beta^2t^2)}}, \quad \text{............(12)}$$

where
$$C = \iint \zeta(\xi, \eta, 0)\, d\xi\, d\eta\,; \quad \text{......................(13)}$$

and the result may be verified by substitution.

* *Arkiv för Matematik, Astronomi och Fysik*, Upsala, Bd. VII. No. 15 (1911).

"The curves $\zeta=$const. constitute a system of coaxal and similar ellipses, whose centre at $t=0$ coincides with the point ξ, η, and then moves with the velocity $\beta\eta$ parallel to the x-axis. For very small values of t the eccentricity of the ellipse is very small and the angle which the major axis makes with the x-axis is about 45°. With increasing t this angle becomes smaller. At the same time the eccentricity becomes larger. For infinitely great values of t, the angle becomes infinitely small and the eccentricity infinitely great."

When $\beta=0$ in (12), we fall back on Fourier's solution. Without loss of generality we may suppose $\xi=0$, $\eta=0$, and then $(r^2=x^2+y^2)$

$$\zeta(x, y, t)=\frac{Ce^{-r^2/4\nu t}}{4\pi\nu t}, \quad \text{......................}(14)$$

representing the diffusion of heat, or vorticity, in two dimensions. It may be worth while to notice the corresponding tangential velocity in the hydrodynamical problem. If ψ be the stream-function,

$$2\zeta=\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right),$$

so that

$$r\frac{d\psi}{dr}=\frac{C}{\pi}(1-e^{-r^2/4\nu t}), \quad \text{........................}(15)$$

the constant of integration being determined from the known value of $d\psi/dr$ when $r=\infty$. When r is small (15) gives

$$\frac{d\psi}{dr}=\frac{Cr}{4\pi\nu t}, \quad \text{..............................}(16)$$

becoming finite when $r=0$ so soon as t is finite.

At time t the greatest value of $d\psi/dr$ occurs when

$$r^2=1{\cdot}256\times 4\nu t. \quad \text{........................}(17)$$

On the basis of his solution Oseen treats the problem of the stability of the shearing motion between two parallel planes and he arrives at the conclusion, in accordance with Kelvin, that the motion is stable for infinitesimal disturbances. For this purpose he considers "the specially unfavourable case" where the distance between the planes is infinitely great. I cannot see myself that Oseen has proved his point. It is doubtless true that a great distance between the planes is unfavourable to stability, but to arrive at a sure conclusion there must be no limitation upon the character of the infinitesimal disturbance, whereas (as it appears to me) Oseen assumes that the disturbance does not sensibly reach the walls. The simultaneous evanescence at the walls of both velocity-components of an otherwise sensible disturbance would seem to be of the essence of the question.

It may be added that Oseen is disposed to refer the instability observed in practice not merely to the square of the disturbance neglected in (6), but also to the inevitable unevenness of the walls.

We may perhaps convince ourselves that the infinitesimal disturbances of (6), with $U = \beta y$, tend to die out by an argument on the following lines, in which it may suffice to consider the operation of a single wall. The argument could, I think, be extended to both walls, but the statement is more complicated. When there is but one wall, we may as well fix ideas by supposing that the wall is at rest (at $y = 0$).

The difficulty of the problem arises largely from the circumstance that the operation of the wall cannot be imitated by the introduction of imaginary vorticities on the further side, allowing the fluid to be treated as uninterrupted. We may indeed in this way satisfy *one* of the necessary conditions. Thus if corresponding to every real vorticity at a point on the positive side we introduce the opposite vorticity at the image of the point in the plane $y = 0$, we secure the annulment in an unlimited fluid of the velocity-component v parallel to y, but the component u, parallel to the flow, remains finite. In order further to annul u, it is in general necessary to introduce new vorticity at $y = 0$. The vorticities on the positive side are not wholly arbitrary.

Let us suppose that initially the only (additional) vorticity in the interior of the fluid is at A, and that this vorticity is clockwise, or positive, like that of the undisturbed motion (fig. 2). If this existed alone, there would be of necessity a finite velocity u along the wall in its neighbourhood. In order

Fig. 2.

Fig. 3.

to satisfy the condition $u = 0$, there must be instantaneously introduced at the wall a negative vorticity of an amount sufficient to give compensation. To this end the local intensity must be inversely as the distance from A and as the sine of the angle between this distance and the wall (Helmholtz). As we have seen these vorticities tend to diffuse and in addition to move with the velocity of the fluid, those near the wall slowly and those arising from A more quickly. As A is carried on, new negative vorticities are developed at those parts of the wall which are being approached. At the other end the vorticities near the wall become excessive and must be compensated. To effect this, new *positive* vorticity must be developed at the wall, whose diffusion over short distances rapidly annuls the negative so far

as may be required. After a time, dependent upon its distance, the vorticity arising from A loses its integrity by coming into contact with the negative diffusing from the wall and thus suffers diminution. It seems evident that the end can only be the annulment of all the additional vorticity and restoration of the undisturbed condition. So long as we adhere to the suppositions of equation (6), the argument applies equally well to an original negative vorticity at A, and indeed to any combination of positive and negative vorticities, however distributed.

It is interesting to inquire how this argument would be affected by the retention in (5) of the additional velocities u, v, which are omitted in (6), though a definite conclusion is hardly to be expected. In fig. 2 the negative vorticity which diffuses inwards is subject to a backward motion due to the vorticity at A in opposition to the slow forward motion previously spoken of. And as A passes on, this negative vorticity in addition to the diffusion is also convected inwards in virtue of the component velocity v due to A. The effect is thus a continued passage inwards behind A of negative vorticity, which tends to neutralize in this region the original constant vorticity (Z). When the additional vorticity at A is negative (fig. 3), the convection behind A acts in opposition to diffusion, and thus the positive developed near the wall remains closer to it, and is more easily absorbed as A passes on. It is true that in front of A there is a convection of positive inwards; but it would seem that this would lead to a more rapid annulment of A itself; and that upon the whole the tendency is for the effect of fig. 2 to preponderate. If this be admitted, we may perhaps see in it an explanation of the diminution of vorticity as we recede from a wall observed in certain circumstances. But we are not in a position to decide whether or not a disturbance dies down. By other reasoning (Reynolds, Orr) we know that it will do so if β be small enough in relation to the other elements of the problem, viz. the distance between the walls and the kinematic viscosity ν.

A precise formulation of the problem for free infinitesimal disturbances was made by Orr (1907). We suppose that ζ and v are proportional to $e^{int}.e^{i\beta x}$, where $n = p + iq$. If $\nabla^2 v = S$, we have from (6) and (10)

$$\frac{d^2S}{dy^2} = \left\{k^2 - \frac{q}{\nu} + \frac{i}{\nu}(p + k\beta y)\right\} S, \quad \text{.....................(18)}$$

and

$$\frac{d^2v}{dy^2} - k^2 v = S, \quad \text{.................................(19)}$$

with the boundary conditions that $v = 0$, $dv/dy = 0$ at the walls. Orr easily shows that the period-equation takes the form

$$\int S_1 e^{ky}\,dy \,.\int S_2 e^{-ky}\,dy - \int S_1 e^{-ky}\,dy \,.\int S_2 e^{ky}\,dy = 0, \quad \text{.........(20)}$$

where S_1, S_2 are any two independent solutions of (18), and the integrations are extended over the interval between the walls. An equivalent equation was given a little later (1908) independently by Sommerfeld*.

Stability requires that for no value of k shall any of the q's determined by (20) be negative. In his discussion Orr arrives at the conclusion that this condition is satisfied, though he does not claim that his method is rigorous. Another of Orr's results may be mentioned here. He shows that $p + k\beta y$ necessarily changes sign in the interval between the walls.

The stability problem has further been skilfully treated by v. Mises† and by Hopf‡, the latter of whom worked at the suggestion of Sommerfeld, with the result of confirming the conclusions of Kelvin and Orr. Doubtless the reasoning employed was sufficient for the writers themselves, but the statements of it put forward hardly carry conviction to the mere reader. The problem is indeed one of no ordinary difficulty. It may, however, be simplified in one respect, as has been shown by v. Mises. It suffices to prove that q can never be zero, inasmuch as it is certain that in some cases ($\beta = 0$) q is positive.

In this direction it may be possible to go further. When $\beta = 0$, it is easy to show that not merely q, but $q - k^2\nu$, is positive§. According to Hopf, this is true generally. Hence it should suffice to omit $k^2 - q/\nu$ in (18), and then to prove that the S-solutions obtained from the equation so simplified cannot satisfy (20). The functions S_1 and S_2, satisfying the simplified equation

$$\frac{d^2S}{d\eta^2} = i\eta S, \quad\text{.................................}(21)$$

where η is *real*, being a linear function of y with real coefficients, could be completely tabulated by the combined use of ascending and descending series, as explained by Stokes in his paper of 1857‖. At the walls η takes opposite signs.

Although a simpler demonstration is desirable, there can remain (I suppose) little doubt but that the shearing motion is stable for infinitesimal disturbances. It has not yet been proved theoretically that the stability can fail for finite disturbances on the supposition of perfectly smooth walls; but such failure seems probable. We know from the work of Reynolds, Lorentz, and Orr that no failure of stability can occur unless $\beta D^2/\nu > 177$, where D is the distance between the walls, so that βD represents their relative motion.

* *Atti del IV. Congr. intern. dei Math.* Roma (1909).

† *Festschrift H. Weber*, Leipzig (1912), p. 252; *Jahresber. d. Deutschen Math. Ver.* Bd. XXI. p. 241 (1913). The mathematics has a very wide scope.

‡ *Ann. der Physik*, Bd. XLIV. p. 1 (1914).

§ *Phil. Mag.* Vol. XXXIV. p. 69 (1892); *Scientific Papers*, Vol. III. p. 583.

‖ *Camb. Phil. Trans.* Vol. X. p. 106; *Math. and Phys. Papers*, Vol. IV. p. 77. This appears to have long preceded the work of Hankel. I may perhaps pursue the line of inquiry here suggested.

389.

NOTE ON THE FORMULA FOR THE GRADIENT WIND.

[*Advisory Committee for Aeronautics. Reports and Memoranda.* No. 147. January, 1915.]

An instantaneous derivation of the formula for the "gradient wind" has been given by Gold*. "For the steady horizontal motion of air along a path whose radius of curvature is r, we may write directly the equation

$$\frac{(\omega r \sin\lambda \pm v)^2}{r} = \frac{1}{\rho}\frac{dp}{dr} + \frac{(\omega r \sin\lambda)^2}{r},$$

expressing the fact that the part of the centrifugal force arising from the motion of the wind is balanced by the effective gradient of pressure.

"In the equation p is atmospheric pressure, ρ density, v velocity of moving air, λ is latitude, and ω is the angular velocity of the earth about its axis." Gold deduces interesting consequences relating to the motion and pressure of air in anti-cyclonic regions†.

But the equation itself is hardly obvious without further explanations, unless we limit it to the case where $\sin\lambda = 1$ (at the pole) and where the relative motion of the air takes place about the same centre as the earth's rotation. I have thought that it may be worth while to take the problem avowedly in two dimensions, but without further restriction upon the character of the relative steady motion.

The axis of rotation is chosen as axis of z. The axes of x and y being supposed to rotate in their own plane with angular velocity ω, we denote by u, v, the velocities at time t, relative to these axes, of the particle which then occupies the position x, y. The actual velocities of the same particle, parallel to the instantaneous positions of the axes, will be $u - \omega y$, $v + \omega x$, and the accelerations in the same directions will be

$$\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} - 2\omega v - \omega^2 x$$

* *Proc. Roy. Soc.* Vol. LXXX A. p. 436 (1908).

† See also Shaw's *Forecasting Weather*, Chapter II.

and

$$\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+2\omega u-\omega^2 y\text{*}.$$

Since the relative motion is supposed to be steady, du/dt, dv/dt disappear, and the dynamical equations are

$$\frac{1}{\rho}\frac{dp}{dx}=\omega^2 x+2\omega v-u\frac{du}{dx}-v\frac{du}{dy}, \quad \text{.....................(1)}$$

$$\frac{1}{\rho}\frac{dp}{dy}=\omega^2 y-2\omega u-u\frac{dv}{dx}-v\frac{dv}{dy}. \quad \text{.....................(2)}$$

The velocities u, v may be expressed by means of the relative stream-function ψ:

$$u=d\psi/dy, \qquad v=-d\psi/dx.$$

Equations (1), (2) then become

$$\frac{1}{\rho}\frac{dp}{dx}=\omega^2 x-2\omega\frac{d\psi}{dx}-\frac{1}{2}\frac{d}{dx}\left\{\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2\right\}+\nabla^2\psi\,.\,\frac{d\psi}{dx}, \quad \text{......(3)}$$

$$\frac{1}{\rho}\frac{dp}{dy}=\omega^2 y-2\omega\frac{d\psi}{dy}-\frac{1}{2}\frac{d}{dy}\left\{\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2\right\}+\nabla^2\psi\,.\,\frac{d\psi}{dy}; \quad \text{......(4)}$$

and on integration, if we leave out the part of p independent of the relative motion,

$$\frac{p}{\rho}=-2\omega\psi-\frac{1}{2}\left\{\left(\frac{d\psi}{dx}\right)^2+\left(\frac{d\psi}{dy}\right)^2\right\}+\int\nabla^2\psi\,d\psi, \quad \text{...........(5)}$$

in which by a known theorem $\nabla^2\psi$ is a function of ψ only. If ω be omitted, (5) coincides with the equation given long ago by Stokes†. It expresses p in terms of ψ; but it does not directly allow of the expression of ψ in terms of p, as is required if the data relate to a barometric chart.

We may revert to the more usual form, if in (1) or (3) we take the axis of x perpendicular to the direction of (relative) motion at any point. Then $u=0$, and

$$\frac{1}{\rho}\frac{dp}{dx}=2\omega v+\frac{d\psi}{dx}\frac{d^2\psi}{dy^2}. \quad \text{.......................(6)}$$

But since $d\psi/dy=0$, the curvature at this place of the stream-line ($\psi=$ const.) is

$$\pm\frac{1}{r}=\frac{d^2\psi}{dy^2}\div\frac{d\psi}{dx},$$

and thus

$$\frac{1}{\rho}\frac{dp}{dx}=2\omega v\pm\frac{v^2}{r}, \quad \text{..............................(7)}$$

* Lamb's *Hydrodynamics*, § 206.

† *Camb. Phil. Trans.* Vol. VII. 1842; *Math. and Phys. Papers*, Vol. I. p. 9.

giving the velocity v in terms of the barometric gradient dp/dx by means of a quadratic. As is evident from the case $\omega = 0$, the positive sign in the alternative is to be taken when x and r are drawn in opposite directions.

In (7) r is not derivable from the barometric chart, nor can ψ be determined strictly by means of p. But in many cases it appears that the more important part of p, at any rate in moderate latitudes, is that which depends upon ω, so that approximately from (5)

$$\psi = -p/2\rho\omega. \qquad \text{.............................(8)}$$

Substituting this value of ψ in the smaller terms, we get as a second approximation

$$2\rho\omega \,.\, \psi = -p - \frac{1}{8\omega^2\rho}\left\{\left(\frac{dp}{dx}\right)^2 + \left(\frac{dp}{dy}\right)^2\right\} + \frac{1}{4\omega^2\rho}\int \nabla^2 p \, dp. \qquad \text{......(9)}$$

With like approximation we may identify r in (7) with the radius of curvature of the *isobaric* curve which passes through the point in question.

The interest of these formulæ depends largely upon the fact that the velocity calculated as above from the barometric gradient represents fairly well the wind actually found at a moderate elevation. At the surface the discrepancy is larger, especially over the land, owing doubtless to friction.

390.

SOME PROBLEMS CONCERNING THE MUTUAL INFLUENCE OF RESONATORS EXPOSED TO PRIMARY PLANE WAVES.

[*Philosophical Magazine*, Vol. XXIX. pp. 209—222, 1915.]

RECENT investigations, especially the beautiful work of Wood on "Radiation of Gas Molecules excited by Light"*, have raised questions as to the behaviour of a cloud of resonators under the influence of plane waves of their own period. Such questions are indeed of fundamental importance. Until they are answered we can hardly approach the consideration of *absorption*, viz. the conversion of radiant into thermal energy. The first action is upon the molecule. We may ask whether this can involve on the average an increase of translatory energy. It does not seem likely. If not, the transformation into thermal energy must await collisions.

The difficulties in the way of answering the questions which naturally arise are formidable. In the first place we do not understand what kind of vibration is assumed by the molecule. But it seems desirable that a beginning should be made; and for this purpose I here consider the case of the simple aerial resonator vibrating symmetrically. The results cannot be regarded as even roughly applicable in a quantitative sense to radiation, inasmuch as this type is inadmissible for transverse vibrations. Nevertheless they may afford suggestions.

The action of a simple resonator under the influence of suitably tuned primary aerial waves was considered in *Theory of Sound*, § 319 (1878). The primary waves were supposed to issue from a simple source at a finite distance c from the resonator. With suppression of the time-factor, and at a distance r from their source, they are represented† by the potential

$$\phi = \frac{e^{-ikr}}{r}, \qquad (1)$$

* A convenient summary of many of the more important results is given in the Guthrie Lecture, *Proc. Phys. Soc.* Vol. XXVI. p. 185 (1914).

† A slight change of notation is introduced.

in which $k = 2\pi/\lambda$, and λ is the wave-length; and it appeared that the potential of the secondary waves diverging from the resonator is

$$\psi = \frac{e^{-ikc}}{ikc} \frac{e^{-ikr'}}{r'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

so that

$$4\pi r'^2 \operatorname{Mod}^2 \psi = 4\pi/k^2c^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The left-hand member of (3) may be considered to represent the energy dispersed. At the distance of the resonator

$$\operatorname{Mod}^2 \phi = 1/c^2.$$

If we inquire what area S of primary wave-front propagates the same energy as is dispersed by the resonator, we have

$$S/c^2 = 4\pi/k^2c^2,$$

or

$$S = 4\pi/k^2 = \lambda^2/\pi. \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Equation (4) applies of course to *plane* primary waves, and is then a particular case of a more general theorem established by Lamb*.

It will be convenient for our present purpose to start *de novo* with plane primary waves, still supposing that the resonator is simple, so that we are concerned only with symmetrical terms, of zero order in spherical harmonics.

Taking the place of the resonator as origin and the direction of propagation as initial line, we may represent the primary potential by

$$\phi = e^{ikr\cos\theta} = 1 + ikr\cos\theta - \tfrac{1}{2}k^2r^2\cos^2\theta + \ldots. \quad \ldots\ldots\ldots(5)$$

The potential of the symmetrical waves issuing from the resonator may be taken to be

$$\psi = \frac{ae^{-ikr}}{r} = \frac{a}{r}(1 - ikr + \ldots). \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Since the resonator is supposed to be an ideal resonator, concentrated in a point, r is to be treated as infinitesimal in considering the conditions to be there satisfied. The first of these is that no *work* shall be done at the resonator, and it requires that total pressure and total radial velocity shall be in quadrature. The total pressure is proportional to $d(\phi+\psi)/dt$, or to $i(\phi+\psi)$, and the total radial velocity is $d(\phi+\psi)/dr$. Thus $(\phi+\psi)$ and $d(\phi+\psi)/dr$ must be in the same (or opposite) phases, in other words their *ratio* must be *real*. Now, with sufficient approximation,

$$\phi + \psi = 1 + \frac{a}{r}(1 - ikr), \quad \frac{d(\phi+\psi)}{dr} = -\frac{a}{r^2};$$

so that

$$a^{-1} - ik = \text{real}. \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

* *Camb. Trans.* Vol. XVIII. p. 348 (1899); *Proc. Math. Soc.* Vol. XXXII. p. 11 (1900). The resonator is no longer limited to be *simple*. See also Rayleigh, *Phil. Mag.* Vol. III. p. 97 (1902); *Scientific Papers*, Vol. V. p. 8.

If we write

$$a = Ae^{i\alpha}, \quad 1/a = A^{-1}e^{-i\alpha}, \quad \text{.......................(8)}$$

then

$$A = -k^{-1}\sin\alpha. \quad \text{.........................(9)}$$

So far α is arbitrary, since we have used no other condition than that no work is being done at the resonator. For instance, (9) applies when the source of disturbance is merely the presence at the origin of a small quantity of gas of varied character. The peculiar action of a *resonator* is to make A a maximum, so that $\sin\alpha = \pm 1$, say -1. Then

$$A = 1/k, \quad a = -i/k, \quad \text{.......................(10)}$$

and

$$\psi = -\frac{ie^{-ikr}}{kr}. \quad \text{.............................(11)}$$

As in (3),

$$4\pi r^2 \operatorname{Mod}^2 \psi = 4\pi/k^2 = \lambda^2/\pi, \quad \text{....................(12)}$$

and the whole energy dispersed corresponds to an area of primary wave-front equal to λ^2/π.

The condition of resonance implies a definite relation between $(\phi + \psi)$ and $d(\phi + \psi)/dr$. If we introduce the value of a from (10), we see that this is

$$\frac{\phi + \psi}{d(\phi + \psi)/dr} = \frac{1/a + 1/r - ik}{-1/r^2} = -r; \quad \text{..............(13)}$$

and this is the relation which must hold at a resonator so tuned as to respond to the primary waves, when isolated from all other influences.

The above calculation relates to the case of a single resonator. For many purposes, especially in Optics, it would be desirable to understand the operation of a company of resonators. A strict investigation of this question requires us to consider each resonator as under the influence, not only of the primary waves, but also of the secondary waves dispersed by its neighbours, and in this many difficulties are encountered. If, however, the resonators are not too near one another, or too numerous, they may be supposed to act independently. From (11) it will be seen that the standard of distance is the wave-length.

The action of a number (n) of similar and irregularly situated centres of secondary disturbance has been considered in various papers on the light from the sky*. The phase of the disturbance from a single centre, as it reaches a distant point, depends of course upon this distance and upon the situation of the centre along the primary rays. If all the circumstances are accurately prescribed, we can calculate the aggregate effect at a distant point, and the resultant intensity may be anything between 0 and that corresponding to complete agreement of phase among all the components. But such a calculation would have little significance for our present purpose.

* Compare also "Wave Theory of Light," *Enc. Brit.* Vol. xxiv. (1888), § 4; *Scientific Papers*, Vol. iii. pp. 53, 54.

Owing to various departures from ideal simplicity, *e.g.* want of homogeneity in the primary vibrations, movement of the disturbing centres, the impossibility of observing what takes place at a mathematical point, we are in effect only concerned with the average, and the average intensity is n times that due to a single centre.

In the application to a cloud of acoustic resonators the restriction was necessary that the resonators must not be close compared with λ; otherwise they would react upon one another too much. This restriction may appear to exclude the case of the light from the sky, regarded as due mainly to the molecules of air; but these molecules are not resonators—at any rate as regards visible radiations. We can most easily argue about an otherwise uniform medium disturbed by numerous small obstacles composed of a medium of different quality. There is then no difficulty in supposing the obstacles so small that their mutual reaction may be neglected, even although the average distance of immediate neighbours is much less than a wave-length. When the obstacles are small enough, the whole energy dispersed may be trifling, but it is well to observe that there must be some. No medium can be fully transparent in all directions to plane waves, which is not itself quite uniform. Partial exceptions may occur, *e.g.* when the want of uniformity is a stratification in plane strata. The dispersal then becomes a regular reflexion, and this may vanish in certain cases, even though the changes of quality are sudden (black in Newton's rings)*. But such transparency is limited to certain directions of propagation.

To return to resonators: when they may be close together, we have to consider their mutual reaction. For simplicity we will suppose that they all lie on the same primary wave-front, so that as before in the neighbourhood of each resonator we may take

$$\phi = 1, \quad d\phi/dr = 0. \qquad \text{.........................(14)}$$

Further, we suppose that all the resonators are similarly situated as regards their neighbours, *e.g.*, that they lie at the angular points of a regular polygon. The waves diverging from each have then the same expression, and altogether

$$\psi = a\left\{\frac{e^{-ikr_1}}{r_1} + \frac{e^{-ikr_2}}{r_2} + \dots\right\}, \qquad \text{....................(15)}$$

where r_1, r_2, ... are the distances of the point where ψ is measured from the various resonators, and a is a coefficient to be determined. The whole potential is $\phi + \psi$, and it suffices to consider the state of things at the first resonator. With sufficient approximation

$$\phi + \psi = 1 + \frac{a}{r_1}(1 - ikr_1) + a\Sigma\,\frac{e^{-ikR}}{R}, \qquad \text{................(16)}$$

* See *Proc. Roy. Soc.* Vol. LXXXVI A, p. 207 (1912); [This volume, p. 77].

R being the distance of any other resonator from the first, while (as before)

$$\frac{d(\phi+\psi)}{dr}=-\frac{a}{r_1^2}. \quad \text{.........................(17)}$$

We have now to distinguish two cases. In the first, which is the more important, the tuning of the resonators is such that each singly would respond as much as possible to the primary waves. The ratio of (16) to (17) must then, as we have seen, be equal to $-r_1$, when r_1 is indefinitely diminished. Accordingly

$$\frac{1}{a}=ik-\Sigma\frac{e^{-ikR}}{R}, \quad \text{.........................(18)}$$

which, of course, includes (10). If we write $a=Ae^{i\alpha}$, then

$$A^2=\frac{1/k^2}{\left[\Sigma\frac{\cos kR}{kR}\right]^2+\left[1+\Sigma\frac{\sin kR}{kR}\right]^2}. \quad \text{..............(19)}$$

The other case arises when the resonators are so tuned that the *aggregate* responds as much as possible to the primary waves. We may then proceed as in the investigation for a single resonator. In order that no work may be done at the disturbing centres, $(\phi+\psi)$ and $d(\phi+\psi)/dr$ must be in the same phase, and this requires that

$$\frac{1}{a}+\frac{1}{r_1}-ik+\Sigma\frac{e^{-ikR}}{R}=\text{real},$$

or

$$\frac{1}{a}=\text{real}+ik+i\Sigma\frac{\sin kR}{R}. \quad \text{......................(20)}$$

The condition of maximum resonance is that the real part in (20) shall vanish, so that

$$\frac{1}{a}=ik\left\{1+\Sigma\frac{\sin kR}{kR}\right\} \quad \text{........................(21)}$$

or

$$A=\frac{1/k}{1+\Sigma\frac{\sin kR}{kR}}. \quad \text{...........................(22)}$$

The present value of A^2 is greater than that in (19), as was of course to be expected. In either case the disturbance is given by (15) with the value of a determined by (18), or (21).

The simplest example is when there are only two resonators and the sign of summation may be omitted in (18). In order to reckon the energy dispersed, we may proceed by either of two methods. In the first we consider the value of ψ and its modulus at a great distance r from the resonators. It is evident that ψ is symmetrical with respect to the line R joining the resonators, and if θ be the angle between r and R, $r_1-r_2=R\cos\theta$. Thus

$$r^2 . \operatorname{Mod}^2\psi=A^2\{2+2\cos(kR\cos\theta)\};$$

and on integration over angular space,

$$2\pi r^2 \int_0^\pi \mathrm{Mod}^2 \psi \,.\, \sin\theta \, d\theta = 8\pi A^2 \left\{1 + \frac{\sin kR}{kR}\right\}. \quad \text{.........(23)}$$

Introducing the value of A^2 from (19), we have finally

$$2\pi r^2 \int_0^\pi \mathrm{Mod}^2 \psi \,.\, \sin\theta \, d\theta = \frac{8\pi k^{-2}\left(1 + \dfrac{\sin kR}{kR}\right)}{1 + \dfrac{1}{k^2R^2} + 2\dfrac{\sin kR}{kR}}. \quad \text{.........(24)}$$

If we suppose that kR is large, but still so that R is small compared with r, (24) reduces to $8\pi k^{-2}$ or $2\lambda^2/\pi$. The energy dispersed is then the double of that which would be dispersed by each resonator acting alone; otherwise the mutual reaction complicates the expression.

The greatest interference naturally occurs when kR is small. (24) then becomes $2k^2R^2 \,.\, 2\lambda^2/\pi$, or $16\pi R^2$, in agreement with *Theory of Sound*, § 321. The whole energy dispersed is then much *less* than if there were only one resonator.

It is of interest to trace the influence of distance more closely. If we put $kR = 2\pi m$, so that $R = m\lambda$, we may write (24)

$$S = (2\lambda^2/\pi) \,.\, F, \quad \text{............................(25)}$$

where S is the area of primary wave-front which carries the same energy as is dispersed by the two resonators and

$$F = \frac{2\pi m + \sin(2\pi m)}{2\pi m + (2\pi m)^{-1} + 2\sin(2\pi m)}. \quad \text{..................(26)}$$

If $2m$ is an integer, the sine vanishes and

$$F = \frac{1}{1 + (2\pi m)^{-2}}, \quad \text{.........................(27)}$$

not differing much from unity even when $2m = 1$; and whenever $2m$ is great, F approaches unity.

The following table gives the values of F for values of $2m$ not greater than 2:

$2m$	F	$2m$	F	$2m$	F
0·05	0·0459	0·70	0·7042	1·40	1·266
0·10	0·1514	0·80	0·7588	1·50	1·259
0·20	0·3582	0·90	0·8256	1·60	1·226
0·30	0·4836	1·00	0·9080	1·70	1·159
0·40	0·5583	1·10	1·006	1·80	1·088
0·50	0·6110	1·20	1·113	1·90	1·026
0·60	0·6569	1·30	1·208	2·00	0·975

In the case of two resonators the integration in (23) presents no difficulty; but when there are a larger number, it is preferable to calculate the emission of energy in the dispersed waves from the work which would have to be done to generate them at the resonators (in the absence of primary waves)—a method which entails no integration. We continue to suppose that all the resonators are similarly situated, so that it suffices to consider the work done at one of them—say the first. From (15)

$$\psi = a\left\{\frac{1-ikr}{r} + \Sigma\frac{e^{-ikR}}{R}\right\}, \quad \frac{d\psi}{dr} = -\frac{a}{r^2}.$$

The pressure is proportional to $i\psi$, and the part of it which is in the same phase as $d\psi/dr$ is proportional to

$$a\left\{k + \Sigma\frac{\sin kR}{R}\right\}.$$

Accordingly the work done at each source is proportional to

$$A^2\left\{1 + \Sigma\frac{\sin kR}{kR}\right\}. \qquad \text{.........................(28)}$$

Hence altogether by (19) the energy dispersed by n resonators is that carried by an area S of primary wave-front, where

$$S = \frac{n\lambda^2}{\pi}\,\frac{1+\Sigma\dfrac{\sin kR}{kR}}{\left[\Sigma\dfrac{\cos kR}{kR}\right]^2 + \left[1+\Sigma\dfrac{\sin kR}{kR}\right]^2}, \qquad \text{...........(29)}$$

the constant factor being determined most simply by a comparison with the case of a single resonator, for which $n=1$ and the Σ's vanish. We fall back on (24) by merely putting $n=2$, and dropping the signs of summation, as there is then only one R.

If the tuning is such as to make the effect of the *aggregate* of resonators a maximum, the cosines in (29) are to be dropped, and we have

$$S = \frac{n\lambda^2/\pi}{1+\Sigma\dfrac{\sin kR}{kR}}. \qquad \text{.........................(30)}$$

As an example of (29), we may take 4 resonators at the angular points of a square whose side is b. There are then 3 R's to be included in the summation, of which two are equal to b and one to $b\sqrt{2}$, so that (28) becomes

$$A^2\left\{1 + 2\frac{\sin kb}{kb} + \frac{\sin(kb\sqrt{2})}{kb\sqrt{2}}\right\}. \qquad \text{.................(31)}$$

A similar result may be arrived at from the value of ψ at an infinite distance, by use of the definite integral*

$$\int_0^{\frac{1}{2}\pi} J_0(x \sin\theta)\sin\theta\, d\theta = \frac{\sin x}{x}. \quad \text{.................(32)}$$

As an example where the company of resonators extends to infinity, we may suppose that there is a row of them, equally spaced at distance R. By (18)

$$\frac{1}{a} = ik - 2\left\{\frac{e^{-ikR}}{R} + \frac{e^{-2ikR}}{2R} + \frac{e^{-3ikR}}{3R} + \dots\right\}. \quad \text{...........(33)}$$

The series may be summed. If we write

$$\Sigma = e^{-ix} + \frac{he^{-2ix}}{2} + \frac{h^2 e^{-3ix}}{3} + \dots, \quad \text{.................(34)}$$

where h is real and less than unity, we have

$$\frac{d\Sigma}{dx} = -\frac{i e^{-ix}}{1 - he^{-ix}},$$

and

$$\Sigma = -\frac{1}{h}\log(1 - he^{-ix}), \quad \text{......................(35)}$$

no constant of integration being required, since

$$\Sigma = -h^{-1}\log(1-h) \quad \text{when } x = 0.$$

If now we put $h = 1$,

$$\Sigma = -\log(1 - e^{-ix}) = -\log\left(2\sin\frac{x}{2}\right) + \tfrac{1}{2}i(x - \pi) + 2in\pi. \quad \text{......(36)}$$

Thus

$$\frac{1}{ka} = i - \frac{2}{kR}\left\{-\log\left(2\sin\frac{kR}{2}\right) + \tfrac{1}{2}i(kR - \pi) + 2in\pi\right\}. \quad \text{......(37)}$$

If $kR = 2m\pi$, or $R = m\lambda$, where m is an integer, the logarithm becomes infinite and a tends to vanish†.

When R is very small, a is also very small, tending to

$$a = R \div 2\log(kR). \quad \text{.........................(38)}$$

The longitudinal density of the now approximately linear source may be considered to be a/R, and this tends to vanish. The multiplication of resonators ultimately annuls the effect at a distance. It must be remembered that the tuning of each resonator is supposed to be as for itself alone.

In connexion with this we may consider for a moment the problem in two dimensions of a linear resonator parallel to the primary waves, which responds symmetrically. As before, we may take at the resonator

$$\phi = 1, \quad d\phi/dr = 0.$$

* *Enc. Brit.* l. c. equation (43); *Scientific Papers*, Vol. III. p. 98.

† *Phil. Mag.* Vol. XIV. p. 60 (1907); *Scientific Papers*, Vol. V. p. 409.

As regards ψ, the potential of the waves diverging in two dimensions, we must use different forms when r is small (compared with λ) and when r is large*. When r is small

$$\psi/a = \left(\gamma + \log\frac{ikr}{2}\right)\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2 . 4^2} - \ldots\right\}$$

$$+ \frac{k^2r^2}{2^2} - \frac{k^4r^4}{2^2 . 4^2}(1 + \tfrac{1}{2}) + \frac{k^6r^6}{2^2 . 4^2 . 6^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) - \ldots; \quad \text{.........(39)}$$

and when r is large,

$$\psi/a = -\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}\left\{1 - \frac{1^2}{1 . 8ikr} + \frac{1^2 . 3^2}{1 . 2 . (8ikr)^2} - \ldots\right\}. \quad \text{...(40)}$$

By the same argument as for a point resonator we find, as the condition that no work is done at $r = 0$, that the imaginary part of $1/a$ is $-i\pi/2$. For maximum resonance

$$a \doteq 2i/\pi, \quad \text{...............................(41)}$$

so that at a distance ψ approximates to

$$\psi = -\frac{\sqrt{\lambda}}{\pi\sqrt{r}} e^{-i(kr - \frac{1}{4}\pi)}. \quad \text{.......................(42)}$$

Thus

$$2\pi r . \mathrm{Mod}^2 \psi = \frac{2\lambda}{\pi}, \quad \text{..........................(43)}$$

which expresses the width of primary wave-front carrying the same energy as is dispersed by the linear resonator tuned to maximum resonance.

A subject which naturally presents itself for treatment is the effect of a distribution of point resonators over the whole plane of the primary wave-front. Such a distribution may be either regular or haphazard. A regular distribution, *e.g.* in square order, has the advantage that all the resonators are similarly situated. The whole energy dispersed is then expressed by (29), though the interpretation presents difficulties in general. But even this would not cover all that it is desirable to know. Unless the side of the square (b) is smaller than λ, the waves directly reflected back are accompanied by lateral "spectra" whose directions may be very various. When $b < \lambda$, it seems that these are got rid of. For then not only the infinite lines forming sides of the squares which may be drawn through the points, but *a fortiori* lines drawn obliquely, such as those forming the diagonals, are too close to give spectra. The whole of the effect is then represented by the specular reflexion.

In some respects a haphazard distribution forms a more practical problem, especially in connexion with resonating vapours. But a precise calculation of the averages then involved is probably not easy.

* *Theory of Sound*, § 341.

If we suppose that the scale (b) of the regular structure is very small compared with λ, we can proceed further in the calculation of the regularly reflected wave. Let Q be one of the resonators, O the point in the plane of the resonators opposite to P, at which ψ is required; $OP = x$, $OQ = y$, $PQ = r$. Then if m be the number of resonators per unit area,

$$\psi = 2\pi ma \int_0^\infty y\, dy\, \frac{e^{-ikr}}{r},$$

or since $y\,dy = r\,dr$,

$$\psi = 2\pi ma \int_x^\infty e^{-ikr}\, dr.$$

The integral, as written, is not convergent; but as in the theory of diffraction we may omit the integral at the upper limit, if we exclude the case of a nearly circular boundary. Thus

$$\psi = \frac{2\pi ma}{ik} e^{-ikx}, \quad \text{.......................(44)}$$

and

$$\text{Mod}^2 \psi = \frac{4\pi^2 m^2 A^2}{k^2}. \quad \text{.......................(45)}$$

The value of A^2 is given by (19). We find, with the same limitation as above,

$$\Sigma \frac{\cos kR}{R} = 2\pi m \int_0^\infty \cos kR\, dR = 0,$$

$$\Sigma \frac{\sin kR}{R} = 2\pi m \int_0^\infty \sin kR dR = 2\pi m/k.$$

Thus $A^2 = 1/(k + 2\pi m/k)^2$

and

$$\text{Mod}^2 \psi = \frac{4\pi^2 m^2}{(k^2 + 2\pi m)^2}. \quad \text{.......................(46)}$$

When the structure is very fine compared with λ, k^2 in the denominator may be omitted, and then $\text{Mod}^2 \psi = 1$, that is the regular reflexion becomes total.

The above calculation is applicable in strictness only to resonators arranged in regular order and very closely distributed. It seems not unlikely that a similar result, viz. a nearly total specular reflexion, would ensue even when there are only a few resonators to the square wave-length, and these are in motion, after the manner of gaseous molecules; but this requires further examination.

In the foregoing investigation we have been dealing solely with forced vibrations, executed in synchronism with primary waves incident upon the resonators, and it has not been necessary to enter into details respecting the constitution of the resonators. All that is required is a suitable adjustment to one another of the virtual mass and spring. But it is also of interest to

consider *free* vibrations. These are of necessity subject to damping, owing to the communication of energy to the medium, forthwith propagated away; and their persistence depends upon the nature of the resonator as regards mass and spring, and not merely upon the ratio of these quantities.

Taking first the case of a single resonator, regarded as bounded at the surface of a small sphere, we have to establish the connexion between the motion of this surface and the aerial pressure operative upon it as the result of vibration. We suppose that the vibrations have such a high degree of persistence that we may calculate the pressure as if they were permanent. Thus if ψ be the velocity-potential, we have as before with sufficient approximation

$$\psi/a = \frac{1-ikr}{r}, \quad \frac{1}{a}\frac{d\psi}{dr} = -\frac{1}{r^2};$$

so that, if ρ be the radial displacement of the spherical surface, $d\rho/dt = -a/r^2$, and

$$\psi = -r(1-ikr)\, d\rho/dt. \qquad (47)$$

Again, if σ be the density of the fluid and δp the variable part of the pressure,

$$\delta p = -\sigma d\psi/dt = \sigma r(1-ikr)\, d^2\rho/dt^2, \qquad (48)$$

which gives the pressure in terms of the displacement ρ at the surface of a sphere of small radius r. Under the circumstances contemplated we may use (48) although the vibration slowly dies down according to the law of e^{int}, where n is not wholly real.

If M denotes the "mass" and μ the coefficient of restitution applicable to ρ, the equation of motion is

$$M\frac{d^2\rho}{dt^2} + \mu\rho + 4\pi\sigma r^3(1-ikr)\frac{d^2\rho}{dt^2} = 0, \qquad (49)$$

or if we introduce e^{int} and write M' for $M + 4\pi\sigma r^3$,

$$n^2(-M' + 4\pi\sigma k r^4 . i) + \mu = 0. \qquad (50)$$

Approximately,

$$n = \sqrt{(\mu/M')} . \{1 + i . 2\pi\sigma k r^4/M'\};$$

and if we write $n = p + iq$,

$$p = \sqrt{(\mu/M')}, \quad q = p . 2\pi\sigma k r^4/M'. \qquad (51)$$

If T be the time in which vibrations die down in the ratio of $e : 1$, $T = 1/q$.

If there be a second precisely similar vibrator at a distance R from the first, we have for the potential

$$\psi_2 = -\frac{r^2}{R} e^{-ikR} \frac{d\rho_2}{dt}, \qquad (52)$$

and for the pressure due to it at the surface of the first vibrator

$$\delta p = \frac{\sigma r^2}{R} e^{-ikR} \frac{d^2\rho_2}{dt^2}. \quad \text{.........................(53)}$$

The equation of motion for ρ_1 is accordingly

$$M\frac{d^2\rho_1}{dt^2} + \mu\rho_1 + 4\pi\sigma r^3 \left\{(1-ikr)\frac{d^2\rho_1}{dt^2} + \frac{re^{-ikR}}{R}\frac{d^2\rho_2}{dt^2}\right\} = 0;$$

and that for ρ_2 differs only by the interchange of ρ_1 and ρ_2. Assuming that both ρ_1 and ρ_2 are as functions of the time proportional to e^{int}, we get to determine n

$$n^2\{M' - 4\pi\sigma r^3 . ikr\} - \mu = \pm n^2 . 4\pi\sigma r^4 R^{-1} e^{-ikR},$$

or approximately

$$n = \sqrt{\frac{\mu}{M'}} . \left\{1 + \frac{2\pi\sigma r^4}{RM'}(ikR \pm e^{-ikR})\right\}. \quad \text{...............(54)}$$

If, as before, we take $n = p + iq$,

$$p = \sqrt{\frac{\mu}{M'}} . \left(1 \pm \frac{2\pi\sigma r^4}{RM'}\cos kR\right), \quad \text{.................(55)}$$

$$q = p . \frac{2\pi\sigma r^4}{RM'}(kR \mp \sin kR). \quad \text{.......................(56)}$$

We may observe that the reaction of the neighbour does not disturb the frequency if $\cos kR = 0$, or the damping if $\sin kR = 0$. When kR is small, the damping in one alternative disappears. The two vibrators then execute their movements in opposite phases and nothing is propagated to a distance.

The importance of the disturbance of frequency in (55) cannot be estimated without regard to the damping. The question is whether the two vibrations get out of step *while they still remain considerable.* Let us suppose that there is a relative gain or loss of half a period while the vibration dies down in the ratio of $e : 1$, viz. in the time denoted previously by T, so that

$$(p_1 - p_2)\, T = \pi.$$

Calling the undisturbed values of p and q respectively P and Q, and supposing kR to be small, we have

$$\frac{P}{Q}\frac{4\pi\sigma r^4}{RM'} = \pi,$$

in which $Q/P = 2\pi\sigma k r^4/M'$. According to this standard the disturbance of frequency becomes important only when $kR < 1/\pi$, or R less than λ/π^2. It has been assumed throughout that r is much less than R.

391.

ON THE WIDENING OF SPECTRUM LINES.

[*Philosophical Magazine*, Vol. XXIX. pp. 274—284, 1915.]

MODERN improvements in optical methods lend additional interest to an examination of the causes which interfere with the absolute homogeneity of spectrum lines. So far as we know these may be considered under five heads, and it appears probable that the list is exhaustive:

(i) The translatory motion of the radiating particles in the line of sight, operating in accordance with Doppler's principle.

(ii) A possible effect of the rotation of the particles.

(iii) Disturbance depending on collision with other particles either of the same or of another kind.

(iv) Gradual dying down of the luminous vibrations as energy is radiated away.

(v) Complications arising from the multiplicity of sources in the line of sight. Thus if the light from a flame be observed through a similar one, the increase of illumination near the centre of the spectrum line is not so great as towards the edges, in accordance with the principles laid down by Stewart and Kirchhoff; and the line is effectively widened. It will be seen that this cause of widening cannot act alone, but merely aggravates the effect of other causes.

There is reason to think that in many cases, especially when vapours in a highly rarefied condition are excited electrically, the first cause is the most important. It was first considered by Lippich* and somewhat later independently by myself†. Subsequently, in reply to Ebert, who claimed to have discovered that the high interference actually observed was inconsistent with Doppler's principle and the theory of gases, I gave a more complete

* *Pogg. Ann.* Vol. CXXXIX. p. 465 (1870).

† *Nature*, Vol. VIII. p. 474 (1873); *Scientific Papers*, Vol. I. p. 183.

calculation*, taking into account the variable velocity of the molecules as defined by Maxwell's law, from which it appeared that there was really no disagreement with observation. Michelson compared these theoretical results with those of his important observations upon light from vacuum-tubes and found an agreement which was thought sufficient, although there remained some points of uncertainty.

The same ground was traversed by Schönrock†, who made the notable remark that while the agreement was good for the monatomic gases it failed for diatomic hydrogen, oxygen, and nitrogen; and he put forward the suggestion that in these cases the chemical atom, rather than the usual molecule, was to be regarded as the carrier of the emission-centres. By this substitution, entailing an increase of velocity in the ratio $\sqrt{2}:1$, the agreement was much improved.

While I do not doubt that Schönrock's comparison is substantially correct, I think that his presentation of the theory is confused and unnecessarily complicated by the introduction (in two senses) of the "width of the spectrum line," a quantity not usually susceptible of direct observation. Unless I misunderstand, what he calls the observed width is a quantity not itself observed at all but deduced from the visibility of interference bands by arguments which already assume Doppler's principle and the theory of gases. I do not see what is gained by introducing this quantity. Given the nature of the radiating gas and its temperature, we can calculate from known data the distribution of light in the bands corresponding to any given retardation, and from photometric experience we can form a pretty good judgment as to the maximum retardation at which they should still be visible. This theoretical result can then be compared with a purely experimental one, and an agreement will confirm the principles on which the calculation was founded. I think it desirable to include here a sketch of this treatment of the question on the lines followed in 1889, but with a few slight changes of notation.

The phenomenon of interference in its simplest form occurs when two equal trains of waves are superposed, both trains having the same frequency and one being retarded relatively to the other by a linear retardation X‡. Then if λ denote the wave-length, the aggregate may be represented by

$$\cos nt + \cos (nt - 2\pi X/\lambda) = 2\cos (\pi X/\lambda) \,.\, \cos (nt - \pi X/\lambda). \;\;\ldots\ldots(1)$$

The intensity is given by

$$I = 4\cos^2 (\pi X/\lambda) = 2\{1 + \cos (2\pi X/\lambda)\}. \;\;\ldots\ldots\ldots\ldots\ldots(2)$$

If we regard X as gradually increasing from zero, I is periodic, the maxima (4) occurring when X is a multiple of λ and the minima (0) when X is an odd

* "On the limits to interference when light is radiated from moving molecules," *Phil. Mag.* Vol. XXVII. p. 298 (1889); *Scientific Papers*, Vol. III. p. 258.

† *Ann. der Physik*, Vol. XX. p. 995 (1906).

‡ In the paper of 1889 the retardation was denoted by 2Δ.

multiple of $\frac{1}{2}\lambda$. If bands are visible corresponding to various values of X, the darkest places are absolutely devoid of light, and this remains true however great X may be, that is however high the order of interference.

The above conclusion requires that the light (duplicated by reflexion or otherwise) should have an absolutely definite frequency, *i.e.* should be absolutely homogeneous. Such light is not at our disposal; and a defect of homogeneity will usually entail a limit to interference, as X increases. We are now to consider the particular defect arising in accordance with Doppler's principle from the motion of the radiating particles in the line of sight. Maxwell showed that for gases in temperature equilibrium the number of molecules whose velocities resolved in three rectangular directions lie within the range $d\xi d\eta d\zeta$ must be proportional to

$$e^{-\beta(\xi^2+\eta^2+\zeta^2)}\, d\xi d\eta d\zeta.$$

If ξ be the direction of the line of sight, the component velocities η, ζ are without influence in the present problem. All that we require to know is that the number of molecules for which the component ξ lies between ξ and $\xi + d\xi$ is proportional to

$$e^{-\beta\xi^2}\, d\xi. \quad \text{.................................(3)}$$

The relation of β to the mean (resultant) velocity v is

$$v = \frac{2}{\sqrt{(\pi\beta)}}. \quad \text{.................................(4)}$$

It was in terms of v that my (1889) results were expressed, but it was pointed out that v needs to be distinguished from the velocity of mean square with which the pressure is more directly connected. If this be called v',

$$v' = \sqrt{\left(\frac{3}{2\beta}\right)}, \quad \text{.................................(5)}$$

so that

$$\frac{v}{v'} = \sqrt{\left(\frac{8}{3\pi}\right)} \quad \text{.................................(6)}$$

Again, the relation between the original wave-length Λ and the actual wave-length λ, as disturbed by the motion, is

$$\frac{\Lambda}{\lambda} = 1 + \frac{\xi}{c}, \quad \text{.................................(7)}$$

c denoting the velocity of light. The intensity of the light in the interference bands, so far as dependent upon the molecules moving with velocity ξ, is by (2)

$$dI = 2\left\{1 + \cos\frac{2\pi X}{\Lambda}\left(1 + \frac{\xi}{c}\right)\right\} e^{-\beta\xi^2}\, d\xi, \quad \text{..................(8)}$$

and this is now to be integrated with respect to ξ between the limits $\pm\infty$. The bracket in (8) is

$$1+\cos\frac{2\pi X}{\Lambda}\cos\frac{2\pi X\xi}{\Lambda c}-\sin\frac{2\pi X}{\Lambda}\sin\frac{2\pi X\xi}{\Lambda c}.$$

The third term, being uneven in ξ, contributes nothing. The remaining integrals are included in the well-known formula

$$\int_{-\infty}^{+\infty} e^{-a^2x^2}\cos(2rx)\,dx=\frac{\sqrt{\pi}}{a}e^{-r^2/a^2}.$$

Thus $$I=\frac{2\sqrt{\pi}}{\sqrt{\beta}}\left[1+\cos\frac{2\pi X}{\Lambda}.\operatorname{Exp}\left(-\frac{\pi^2X^2}{c^2\beta\Lambda^2}\right)\right]\text{ ....(9)}$$

The intensity I_1 at the darkest part of the bands is found by making X an odd multiple of $\frac{1}{2}\lambda$, and I_2 the maximum brightness by making X a multiple of λ.

Thus $$\operatorname{Exp}\left(-\frac{\pi^2X^2}{c^2\beta\Lambda^2}\right)=\frac{I_2-I_1}{I_2+I_1}=V,\text{ ....(10)}$$

where V denotes the "visibility" according to Michelson's definition. Equation (10) is the result arrived at in my former paper, and β can be expressed in terms of either the mean velocity v, or preferably of the velocity of mean square v'*.

The next question is what is the smallest value of V for which the bands are recognizable. Relying on photometric experience, I estimated that a relative difference of 5 per cent. between I_1 and I_2 would be about the limit in the case of high interference bands, and I took $V=\cdot025$. Shortly afterwards† I made special experiments upon bands well under control, obtained by means of double refraction, and I found that in this very favourable case the bands were still just distinctly seen when the relative difference between I_1 and I_2 was reduced to 4 per cent. It would seem then that the estimate $V=\cdot025$ can hardly be improved upon. On this basis (10) gives in terms of v

$$\frac{X}{\Lambda}=\frac{2c}{\pi^{3/2}v}\sqrt{(\log_e 40)}=\cdot690\frac{c}{v},\text{ ....(11)}$$

as before. In terms of v' by (6)

$$\frac{X}{\Lambda}=\frac{\sqrt{3}\,.\,c}{\pi\sqrt{2}\,.\,v'}\sqrt{(\log_e 40)}=\cdot749\frac{c}{v'}.\text{ ....(12)}$$

As an example of (12), let us apply it to hydrogen molecules at 0° C. Here $v'=1839\times 10^2$ cm./sec.‡, and $c=3\times 10^{10}$. Thus

$$X/\Lambda=1\cdot222\times 10^5.\text{ ....(13)}$$

* See also *Proc. Roy. Soc.* Vol. LXXVI A. p. 440 (1905); *Scientific Papers*, Vol. V. p. 261.

† *Phil. Mag.* Vol. XXVII. p. 484 (1889); *Scientific Papers*, Vol. III. p. 277.

‡ It seems to be often forgotten that the first published calculation of molecular velocities was that of Joule (*Manchester Memoirs*, Oct. 1848, *Phil. Mag.* ser. 4, Vol. XIV. p. 211).

This is for the hydrogen *molecule*. For the hydrogen *atom* (13) must be divided by $\sqrt{2}$. Thus for absolute temperature T and for radiating centres whose mass is m times that of the hydrogen *atom*, we have

$$\frac{X}{\Lambda} = \frac{1\cdot222 \times \sqrt{(273)} \times 10^5}{\sqrt{2}} \sqrt{\left(\frac{m}{T}\right)} = 1\cdot427 \times 10^6 \sqrt{\left(\frac{m}{T}\right)}. \quad \text{...(14)}$$

In Buisson and Fabry's corresponding formula, which appears to be derived from Schönrock, 1·427 is replaced by the appreciably different number 1·22*.

The above value of X is the retardation corresponding to the *limit* of visibility, taken to be represented by $V = \cdot025$. In Schönrock's calculation the retardation X_1, corresponding to $V = \cdot5$, is considered. In (12), $\sqrt{(\log_e 40)}$ would then be replaced by $\sqrt{(\log_e 2)}$, and instead of (14) we should have

$$\frac{X_1}{\Lambda} = 6\cdot186 \times 10^5 \sqrt{\left(\frac{m}{T}\right)} \quad \text{......(15)}$$

But I do not understand how $V = \cdot5$ could be recognized in practice with any precision.

Although it is not needed in connexion with high interference, we can of course calculate the width of a spectrum line according to any conventional definition. Mathematically speaking, the width is infinite; but if we disregard the outer parts where the intensity is less than *one-half* the maximum the limiting value of ξ by (3) is given by

$$\beta\xi^2 = \log_e 2, \quad \text{......(16)}$$

and the corresponding value of λ by

$$\frac{\lambda - \Lambda}{\Lambda} = \frac{\xi}{c} = \frac{\sqrt{(\log_e 2)}}{c\sqrt{\beta}}. \quad \text{......(17)}$$

Thus, if $\delta\lambda$ denote the half-width of the line according to the above definition,

$$\frac{\delta\lambda}{\Lambda} = \frac{\sqrt{(\cdot6931)}}{c\sqrt{\beta}} = 3\cdot57 \times 10^{-7} \sqrt{\left(\frac{T}{m}\right)}, \quad \text{......(18)}$$

T denoting absolute temperature and m the mass of the particles in terms of that of the hydrogen atom, in agreement with Schönrock.

In the application to particular cases the question at once arises as to what we are to understand by T and m. In dealing with a flame it is natural to take the temperature of the flame as ordinarily understood, but when we pass to the rare vapour of a vacuum-tube electrically excited, the matter is not so simple. Michelson assumed from the beginning that the temperature with which we are concerned is that of the tube itself or not much higher. This view is amply confirmed by the beautiful experiments of Buisson and Fabry†,

* [1916. I understand from M. Fabry that the difference between our numbers has its origin in a somewhat different estimate of the minimum value of V. The French authors admit an allowance for the more difficult conditions under which high interference is observed.]

† *Journ. de Physique*, t. II. p. 442 (1912).

who observed the limit of interference when tubes containing helium, neon, and krypton were cooled in liquid air. Under these conditions bands which had already disappeared at room temperature again became distinct, and the ratios of maximum retardations in the two cases (1·66, 1·60, 1·58) were not much less than the theoretical 1·73 calculated on the supposition that the temperature of the gas is that of the tube. The highest value of X/Λ, in their notation N, hitherto observed is 950,000, obtained from krypton in liquid air. With all three gases the agreement at room temperature between the observed and calculated values of N is extremely good, but as already remarked their theoretical numbers are a little lower than mine (14). We may say not only that the observed effects are accounted for almost completely by Doppler's principle and the theory of gases, but that the temperature of the emitting gas is not much higher than that of the containing tube.

As regards m, no question arises for the inert monatomic gases. In the case of hydrogen Buisson and Fabry follow Schönrock in taking the atom rather than the molecule as the moving source, so that $m = 1$; and further they find that this value suits not only the lines of the first spectrum of hydrogen but equally those of the second spectrum whose origin has sometimes been attributed to impurities or aggregations.

In the case of sodium, employed in a vacuum-tube, Schönrock found a fair agreement with the observations of Michelson, on the assumption that the *atom* is in question. It may be worth while to make an estimate for the D lines from soda in a Bunsen flame. Here $m = 23$, and we may perhaps take T at 2500. These data give in (14) as the maximum number of bands

$$X/\Lambda = 137{,}000.$$

The number of bands actually seen is very dependent upon the amount of soda present. By reducing this Fizeau was able to count 50,000 bands, and it would seem that this number cannot be much increased*, so that observation falls very distinctly behind calculation†. With a large supply of soda the number of bands may drop to two or three thousand, or even further.

The second of the possible causes of loss of homogeneity enumerated above, viz. *rotation* of the emitting centres, was briefly discussed many years ago in a letter to Michelson‡, where it appeared that according to the views then

* "Interference Bands and their Applications," *Nature*, Vol. XLVIII. p. 212 (1893); *Scientific Papers*, Vol. IV. p. 59. The parallel plate was a layer of water superposed upon mercury. An enhanced illumination may be obtained by substituting nitro-benzol for water, and the reflexions from the mercury and oil may be balanced by staining the latter with aniline blue. But a thin layer of nitro-benzol takes a surprisingly long time to become level.

† Smithells (*Phil. Mag.* Vol. XXXVII. p. 245, 1894) argues with much force that the actually operative parts of the flame may be at a much higher temperature (if the word may be admitted) than is usually supposed, but it would need an almost impossible allowance to meet the discrepancy. The chemical questions involved are very obscure. The coloration with soda appears to require the presence of oxygen (Mitcherlich, Smithells).

‡ *Phil. Mag.* Vol. XXXIV. p. 407 (1892); *Scientific Papers*, Vol. IV. p. 15.

widely held this cause should be more potent than (i). The transverse vibrations emitted from a luminous source cannot be uniform in all directions, and the effect perceived in a fixed direction from a rotating source cannot in general be simple harmonic. In illustration it may suffice to mention the case of a bell vibrating in four segments and rotating about the axis of symmetry. The sound received by a stationary observer is intermittent and therefore not homogeneous. On the principle of equipartition of energy between translatory and rotatory motions, and from the circumstance that the dimensions of molecules are much less than optical wave-lengths, it followed that the loss of homogeneity from (ii) was much greater than from (i). I had in view diatomic molecules—for at that time mercury vapour was the only known exception; and the specific heats at ordinary temperatures showed that two of the possible three rotations actually occurred in accordance with equipartition of energy. It is now abundantly clear that the widening of spectrum lines at present under consideration does not in fact occur; and the difficulty that might be felt is largely met when we accept Schönrock's supposition that the radiating centres are in all cases monatomic. Still there are questions remaining behind. Do the atoms rotate, and if not, why not? I suppose that the quantum theory would help here, but it may be noticed that the question is not merely of acquiring rotation. A permanent rotation, not susceptible of alteration, should apparently make itself felt. These are problems relating to the constitution of the atom and the nature of radiation, which I do not venture further to touch upon.

The third cause of widening is the disturbance of free vibration due to encounters with other bodies. That something of this kind is to be expected has long been recognized, and it would seem that the widening of the D lines when more than a very little soda is present in a Bunsen flame can hardly be accounted for otherwise. The simplest supposition open to us is that an entirely fresh start is made at each collision, so that we have to deal with a series of regular vibrations limited at both ends. The problem thus arising has been treated by Godfrey* and by Schönrock†. The Fourier analysis of the limited train of waves of length r gives for the intensity of various parts of the spectrum line

$$k^{-2}\sin^2(\pi rk), \qquad\qquad (19)$$

where k is the reciprocal of the wave-length, measured from the centre of the line. In the application to radiating vapours, integrations are required with respect to r.

Calculations of this kind serve as illustrations; but it is not to be supposed that they can represent the facts at all completely. There must surely

* *Phil. Trans.* A. Vol. CXCV. p. 346 (1899). See also *Proc. Roy. Soc.* Vol. LXXVI. A. p. 440 (1905); *Scientific Papers*, Vol. v. p. 257.

† *Ann. der Physik*, Vol. XXII. p. 209 (1907).

be encounters of a milder kind where the free vibrations are influenced but yet not in such a degree that the vibrations after the encounter have no relation to the previous ones. And in the case of flames there is another question to be faced: Is there no distinction in kind between encounters first of two sodium atoms and secondly of one sodium atom and an atom say of nitrogen? The behaviour of soda flames shows that there is. Otherwise it seems impossible to explain the great effect of relatively very small additions of soda in presence of large quantities of other gases. The phenomena suggest that the failure of the least coloured flames to give so high an interference as is calculated from Doppler's principle may be due to encounters with other gases, but that the rapid falling off when the supply of soda is increased is due to something special. This might be of a quasi-chemical character, *e.g.* temporary associations of atoms; or again to vibrators in close proximity putting one another out of tune. In illustration of such effects a calculation has been given in the previous paper*. It is in accordance with this view that, as Gouy found, the emission of light tends to increase as the square root of the amount of soda present.

We come now to cause (iv). Although it is certain that this cause must operate, we are not able at the present time to point to any experimental verification of its influence. As a theoretical illustration "we may consider the analysis by Fourier's theorem of a vibration in which the amplitude follows an exponential law, rising from zero to a maximum and afterwards falling again to zero. It is easily proved that

$$e^{-a^2x^2}\cos rx = \frac{1}{2a\sqrt{\pi}}\int_0^\infty du\cos ux\,\{e^{-(u-r)^2/4a^2}+e^{-(u+r)^2/4a^2}\},\quad \ldots(20)$$

in which the second member expresses an aggregate of trains of waves, each individual train being absolutely homogeneous. If a be small in comparison with r, as will happen when the amplitude on the left varies but slowly, $e^{-(u+r)^2/4a^2}$ may be neglected, and $e^{-(u-r)^2/4a^2}$ is sensible only when u is very nearly equal to r"†.

An analogous problem, in which the vibration is represented by $e^{-at}\sin bt$, has been treated by Garbasso‡. I presume that the form quoted relates to positive values of t and that for negative values of t it is to be replaced by zero. But I am not able to confirm Garbasso's formula§.

As regards the fifth cause of (additional) widening enumerated at the beginning of this paper, the case is somewhat similar to that of the fourth. It must certainly operate, and yet it does not appear to be important in practice. In such rather rough observations as I have made, it seems to make no

* *Phil. Mag. supră*, p. 209. [This volume, Art. 390.]

† *Phil. Mag.* Vol. XXXIV. p. 407 (1892); *Scientific Papers*, Vol. IV. p. 16.

‡ *Ann. der Physik*, Vol. XX. p. 848 (1906).

§ Possibly the sign of a is supposed to change when t passes through zero. But even then what are perhaps misprints would need correction.

great difference whether two surfaces of a Bunsen soda flame (front and back) are in action or only one. If the supply of soda to each be insufficient to cause dilatation, the multiplication of flames in line (3 or 4) has no important effect either upon the brightness or the width of the lines. Actual measures, in which no high accuracy is needed, would here be of service.

The observations referred to led me many years ago to make a very rough comparison between the light actually obtained from a nearly undilated soda line and that of the corresponding part of the spectrum from a black body at the same temperature as the flame. I quote it here rather as a suggestion to be developed than as having much value in itself. Doubtless, better data are now available.

How does the intrinsic brightness of a just undilated soda flame compare with the total brightness of a black body at the temperature of the flame? As a source of light Violle's standard, viz. one sq. cm. of just melting platinum, is equal to about 20 candles. The candle presents about 2 sq. cm. of area, so that the radiating platinum is about 40 times as bright. Now platinum is not a black body and the Bunsen flame is a good deal hotter than the melting metal. I estimated (and perhaps under estimated) that a factor of 5 might therefore be introduced, making the black body at flame temperature 200 times as bright as the candle.

To compare with a candle a soda flame of which the D-lines were just beginning to dilate, I reflected the former nearly perpendicularly from a single glass surface. The soda flame seemed about half as bright. At this rate the intrinsic brightness of the flame was $\frac{1}{2} \times \frac{1}{25} = \frac{1}{50}$ of that of the candle, and accordingly $\frac{1}{10,000}$ of that of the black body.

The black body gives a continuous spectrum. What would its brightness be when cut down to the narrow regions occupied by the D-lines? According to Abney's measures the brightness of that part of sunlight which lies between the D's would be about $\frac{1}{250}$ of the whole. We may perhaps estimate the region actually covered by the soda lines as $\frac{1}{25}$ of this. At this rate we should get

$$\frac{1}{25} \times \frac{1}{250} = \frac{1}{6250},$$

as the fraction of the whole radiation of the black body which has the wave-lengths of the soda lines. The actual brightness of a soda flame is thus of the same order of magnitude as that calculated for a black body when its spectrum is cut down to that of the flame, and we may infer that the light of a powerful soda flame is due much more to the widening of the spectrum lines than to an increased brightness of their central parts.

392.

THE PRINCIPLE OF SIMILITUDE.

[*Nature*, Vol. xcv. pp. 66—68, March, 1915.]

I HAVE often been impressed by the scanty attention paid even by original workers in physics to the great principle of similitude. It happens not infrequently that results in the form of "laws" are put forward as novelties on the basis of elaborate experiments, which might have been predicted *a priori* after a few minutes' consideration. However useful verification may be, whether to solve doubts or to exercise students, this seems to be an inversion of the natural order. One reason for the neglect of the principle may be that, at any rate in its applications to particular cases, it does not much interest mathematicians. On the other hand, engineers, who might make much more use of it than they have done, employ a notation which tends to obscure it. I refer to the manner in which gravity is treated. When the question under consideration depends essentially upon gravity, the symbol of gravity (g) makes no appearance, but when gravity does not enter into the question at all, g obtrudes itself conspicuously.

I have thought that a few examples, chosen almost at random from various fields, may help to direct the attention of workers and teachers to the great importance of the principle. The statement made is brief and in some cases inadequate, but may perhaps suffice for the purpose. Some foreign considerations of a more or less obvious character have been invoked in aid. In using the method practically, two cautions should be borne in mind. First, there is no prospect of determining a numerical coefficient from the principle of similarity alone; it must be found, if at all, by further calculation, or experimentally. Secondly, it is necessary as a preliminary step to specify clearly *all* the quantities on which the desired result may reasonably be supposed to depend, after which it may be possible to drop one or more if further consideration shows that in the circumstances they cannot enter. The following, then, are some conclusions, which may be arrived at by this method:

Geometrical similarity being presupposed here as always, how does the strength of a bridge depend upon the linear dimension and the force of gravity?

In order to entail the same strains, the force of gravity must be inversely as the linear dimension. Under a given gravity the larger structure is the weaker.

The velocity of propagation of periodic waves on the surface of deep water is as the square root of the wave-length.

The periodic time of liquid vibration under gravity in a deep cylindrical vessel of any section is as the square root of the linear dimension.

The periodic time of a tuning-fork, or of a Helmholtz resonator, is directly as the linear dimension.

The intensity of light scattered in an otherwise uniform medium from a small particle of different refractive index is inversely as the fourth power of the wave-length.

The resolving power of an object-glass, measured by the reciprocal of the angle with which it can deal, is directly as the diameter and inversely as the wave-length of the light.

The frequency of vibration of a globe of liquid, vibrating in any of its modes under its own gravitation, is independent of the diameter and directly as the square root of the density.

The frequency of vibration of a drop of liquid, vibrating under capillary force, is directly as the square root of the capillary tension and inversely as the square root of the density and as the $1\frac{1}{2}$ power of the diameter.

The time-constant (*i.e.* the time in which a current falls in the ratio $e:1$) of a linear conducting electric circuit is directly as the inductance and inversely as the resistance, measured in electro-magnetic measure.

The time-constant of circumferential electric currents in an infinite conducting cylinder is as the square of the diameter.

In a gaseous medium, of which the particles repel one another with a force inversely as the nth power of the distance, the viscosity is as the $(n+3)/(2n-2)$ power of the absolute temperature. Thus, if $n=5$, the viscosity is proportional to temperature.

Eiffel found that the resistance to a sphere moving through air changes its character somewhat suddenly at a certain velocity. The consideration of viscosity shows that the critical velocity is inversely proportional to the diameter of the sphere.

If viscosity may be neglected, the mass (M) of a drop of liquid, delivered slowly from a tube of diameter (a), depends further upon (T) the capillary tension, the density (σ), and the acceleration of gravity (g). If these data suffice, it follows from similarity that

$$M = \frac{Ta}{g} \cdot F\left(\frac{T}{g\sigma a^2}\right),$$

where F denotes an arbitrary function. Experiment shows that F varies but little and that within somewhat wide limits it may be taken to be 3·8. Within these limits Tate's law that M varies as a holds good.

In the Æolian harp, if we may put out of account the compressibility and the viscosity of the air, the pitch (n) is a function of the velocity of the wind (v) and the diameter (d) of the wire. It then follows from similarity that the pitch is directly as v and inversely as d, as was found experimentally by Strouhal. If we include viscosity (ν), the form is

$$n = v/d \, . \, f(\nu/vd),$$

where f is arbitrary.

As a last example let us consider, somewhat in detail, Boussinesq's problem of the steady passage of heat from a good conductor immersed in a stream of fluid moving (at a distance from the solid) with velocity v. The fluid is treated as incompressible and for the present as inviscid, while the solid has always the same *shape* and presentation to the stream. In these circumstances the total heat (h) passing in unit time is a function of the linear dimension of the solid (a), the temperature-difference (θ), the stream-velocity (v), the capacity for heat of the fluid per unit volume (c), and the conductivity (κ). The density of the fluid clearly does not enter into the question. We have now to consider the "dimensions" of the various symbols.

Those of a are	$(\text{Length})^1$,
" " v "	$(\text{Length})^1 (\text{Time})^{-1}$,
" " θ "	$(\text{Temperature})^1$,
" " c "	$(\text{Heat})^1 (\text{Length})^{-3} (\text{Temp.})^{-1}$,
" " κ "	$(\text{Heat})^1 (\text{Length})^{-1} (\text{Temp.})^{-1} (\text{Time})^{-1}$,
" " h "	$(\text{Heat})^1 (\text{Time})^{-1}$.

Hence if we assume

$$h = a^x \theta^y v^z c^u \kappa^v,$$

we have

$$\begin{array}{ll} \text{by heat} & 1 = u + v, \\ \text{by temperature} & 0 = y - u - v, \\ \text{by length} & 0 = x + z - 3u - v, \\ \text{by time} & -1 = -z - v; \end{array}$$

so that

$$h = \kappa a \theta \left(\frac{avc}{\kappa}\right)^z.$$

Since z is undetermined, any number of terms of this form may be combined, and all that we can conclude is that

$$h = \kappa a \theta \, . \, F(avc/\kappa),$$

where F is an arbitrary function of the one variable avc/κ. An important particular case arises when the solid takes the form of a cylindrical wire of any section, the length of which is perpendicular to the stream. In strictness similarity requires that the length l be proportional to the linear dimension of the section b; but when l is relatively very great h must become proportional to l and a under the functional symbol may be replaced by b. Thus

$$h = \kappa l \theta \,.\, F(bvc/\kappa).$$

We see that in all cases h is proportional to θ, and that for a given fluid F is constant provided v be taken inversely as a or b.

In an important class of cases Boussinesq has shown that it is possible to go further and actually to determine the form of F. When the layer of fluid which receives heat during its passage is very thin, the flow of heat is practically in one dimension and the circumstances are the same as when the plane boundary of a uniform conductor is suddenly raised in temperature and so maintained. From these considerations it follows that F varies as $v^{\frac{1}{2}}$, so that in the case of the wire

$$h \propto l\theta \,.\, \sqrt{(bvc/\kappa)},$$

the remaining constant factor being dependent upon the shape and purely numerical. But this development scarcely belongs to my present subject.

It will be remarked that since viscosity is neglected, the fluid is regarded as flowing past the surface of the solid with finite velocity, a serious departure from what happens in practice. If we include viscosity in our discussion, the question is of course complicated, but perhaps not so much as might be expected. We have merely to include another factor, ν^w, where ν is the kinematic viscosity of dimensions $(\text{Length})^2\ (\text{Time})^{-1}$, and we find by the same process as before

$$h = \kappa a \theta \,.\, \left(\frac{avc}{\kappa}\right)^z \,.\, \left(\frac{c\nu}{\kappa}\right)^w.$$

Here z and w are both undetermined, and the conclusion is that

$$h = \kappa a \theta \,.\, F\left\{\frac{avc}{\kappa},\ \frac{c\nu}{\kappa}\right\},$$

where F is an arbitrary function of the *two* variables avc/κ and $c\nu/\kappa$. The latter of these, being the ratio of the two diffusivities (for momentum and for temperature), is of no dimensions; it appears to be constant for a given kind of gas, and to vary only moderately from one gas to another. If we may assume the accuracy and universality of this law, $c\nu/\kappa$ is a merely numerical constant, the same for all gases, and may be omitted, so that h reduces to the forms already given when viscosity is neglected altogether, F being again a function of a single variable, avc/κ or bvc/κ. In any case F is constant for a given fluid, provided v be taken inversely as a or b.

[*Nature*, Vol. XCV. p. 644, Aug. 1915.]

The question raised by Dr Riabouchinsky (*Nature*, July 29, p. 105)* belongs rather to the logic than to the use of the principle of similitude, with which I was mainly concerned. It would be well worthy of further discussion. The conclusion that I gave follows on the basis of the usual Fourier equation for the conduction of heat, in which heat and temperature are regarded as *sui generis*. It would indeed be a paradox if further knowledge of the nature of heat afforded by molecular theory put us in a worse position than before in dealing with a particular problem. The solution would seem to be that the Fourier equations embody something as to the nature of heat and temperature which is ignored in the alternative argument of Dr Riabouchinsky.

[1917. Reference may be made also to a letter signed J. L. in the same number of *Nature*, and to *Nature*, April 22, 1915. See further Buckingham, *Nature*, Vol. XCVI. p. 396, Dec. 1915. Mr Buckingham had at an earlier date (Oct. 1914) given a valuable discussion of the whole theory (*Physical Review*, Vol. IV. p. 345), and further questions have been raised in the same Review by Tolman.

As a variation of the last example, we may consider the passage of heat between two infinite parallel plane surfaces maintained at fixed temperatures differing by θ, when the intervening space is occupied by a stream of incompressible viscous fluid (*e.g.* water) of mean velocity v. In a uniform regime the heat passing across is proportional to the time and to the area considered; but in many cases the uniformity is not absolute and it is necessary to take the *mean* passage over either a large enough area or a long enough time. On this understanding there is a definite quantity h', representing the passage of heat per unit area and per unit time.

If there be no stream ($v = 0$), or in any case if the kinematic viscosity (ν) is infinite, we have

$$h' = \kappa\theta/a,$$

a being the distance between the surfaces, since then the motion, if any, takes place in plane strata. But when the velocity is high enough, or the viscosity low enough, the motion becomes *turbulent*, and the flow of heat may be greatly augmented. With the same reasoning and with the same notation as before we have

$$h' = \frac{\kappa\theta}{a} \cdot F\left(\frac{avc}{\kappa}, \frac{c\nu}{\kappa}\right).$$

* "In *Nature* of March 18, Lord Rayleigh gives this formula $h = \kappa a\theta \,.\, F(avc/\kappa)$, considering heat, temperature, length, and time as four 'independent' units. If we suppose that only three of these quantities are really independent, we obtain a different result. For example, if the temperature is defined as the mean kinetic energy of the molecules, the principle of similarity allows us only to affirm that $h = \kappa a\theta \,.\, F(v/\kappa a^2, ca^3)$."

or which comes to the same

$$h' = \frac{\kappa\theta}{a} \cdot F_1\left(\frac{av}{\nu}, \frac{c\nu}{\kappa}\right),$$

F, F_1 being arbitrary functions of two variables. And, as we have seen, $F(0, c\nu/\kappa) = 1$.

For a given fluid $c\nu/\kappa$ is constant and may be omitted. Dynamical similarity is attained when av is kept constant, so that a complete determination of F, experimentally or otherwise, does not require a variation of *both* a and v. There is advantage in retaining a constant; for if a varies, geometrical similarity demands that any roughnesses shall be in proportion.

It should not be overlooked that in the above argument, c, κ, ν are treated as constants, whereas they would really vary with the temperature. The assumption is completely justified only when the temperature difference θ is very small.

Another point calls for attention. The régime ultimately established may in some cases depend upon the initial condition. Reynolds' observations suggest that with certain values of av/ν the simple stratified motion once established may persist; but that the introduction of disturbances exceeding a certain amount may lead to an entirely different (turbulent) régime. Over part of the range F would have double values.

It would be of interest to know what F becomes when av tends to infinity. It seems probable that F too becomes infinite. but perhaps very slowly.]

393.

DEEP WATER WAVES, PROGRESSIVE OR STATIONARY, TO THE THIRD ORDER OF APPROXIMATION.

[*Proceedings of the Royal Society*, A, Vol. XCI. pp. 345—353, 1915.]

As is well known, the form of periodic waves progressing over deep water *without change of type* was determined by Stokes* to a high degree of approximation. The wave-length (λ) in the direction of x being 2π and the velocity of propagation unity, the form of the surface is given by

$$y = a\cos(x-t) - \tfrac{1}{2}a^2\cos 2(x-t) + \tfrac{3}{8}a^3\cos 3(x-t), \quad \text{.........(1)}$$

and the corresponding gravity necessary to maintain the motion by

$$g = 1 - a^2. \quad \text{...............................(2)}$$

The generalisation to other wave-lengths and velocities follows by "dimensions."

These and further results for progressive waves of permanent type are most easily arrived at by use of the stream-function on the supposition that the waves are reduced to rest by an opposite motion of the water as a whole, when the problem becomes one of steady motion†. My object at present is to extend the scope of the investigation by abandoning the initial restriction to progressive waves of permanent type. The more general equations may then be applied to progressive waves as a particular case, or to stationary waves in which the principal motion is proportional to a simple circular function of the time, and further to ascertain what occurs when the conditions necessary for the particular cases are not satisfied. Under these circumstances the use of the stream-function loses much of its advantage, and the method followed is akin to that originally adopted by Stokes.

* *Camb. Phil. Trans.* Vol. VIII. p. 441 (1847); *Math. and Phys. Papers*, Vol. I. p. 197.

† *Phil. Mag.* Vol. I. p. 257 (1876); *Scientific Papers*, Vol. I. p. 262. Also *Phil. Mag.* Vol. XXI. p. 183 (1911); [This volume, p. 11].

The velocity-potential ϕ, being periodic in x, may be expressed by the series

$$\phi = \alpha e^{-y} \sin x - \alpha' e^{-y} \cos x + \beta e^{-2y} \sin 2x$$
$$- \beta' e^{-2y} \cos 2x + \gamma e^{-3y} \sin 3x - \gamma' e^{-3y} \cos 3x + \ldots, \quad \ldots(3)$$

where α, α', β, etc., are functions of the time only, and y is measured downwards from mean level. In accordance with (3) the component velocities are given by

$$u = d\phi/dx = e^{-y} (\alpha \cos x + \alpha' \sin x) + 2e^{-2y} (\beta \cos 2x + \beta' \sin 2x) + \ldots$$
$$- v = d\phi/dy = e^{-y} (\alpha \sin x - \alpha' \cos x) + 2e^{-2y} (\beta \sin 2x - \beta' \cos 2x) + \ldots.$$

The density being taken as unity, the pressure equation is

$$p = - d\phi/dt + F + gy - \tfrac{1}{2} (u^2 + v^2), \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

in which F is a function of the time.

In applying (4) we will regard α, α', as small quantities of the first order, while β, β', γ, γ', are small quantities of the second order at most; and for the present we retain only quantities of the second order. β, etc., will then not appear in the expression for $u^2 + v^2$. In fact

$$u^2 + v^2 = e^{-2y} (\alpha^2 + \alpha'^2),$$

and

$$p = - \frac{d\alpha}{dt} e^{-y} \sin x + \frac{d\alpha'}{dt} e^{-y} \cos x - \frac{d\beta}{dt} e^{-2y} \sin 2x + \ldots$$
$$+ gy - \tfrac{1}{2} e^{-2y} (\alpha^2 + \alpha'^2) + F. \quad \ldots(5)$$

The surface conditions are (i) that p be there zero, and (ii) that

$$\frac{Dp}{Dt} = \frac{dp}{dt} + u \frac{dp}{dx} + v \frac{dp}{dy} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The first is already virtually expressed in (5). For the second

$$\frac{dp}{dt} = - \frac{d^2\alpha}{dt^2} e^{-y} \sin x + \frac{d^2\alpha'}{dt^2} e^{-y} \cos x - \ldots - e^{-2y} \left(\alpha \frac{d\alpha}{dt} + \alpha' \frac{d\alpha'}{dt} \right) + F',$$

$$\frac{dp}{dx} = - \frac{d\alpha}{dt} e^{-y} \cos x - \frac{d\alpha'}{dt} e^{-y} \sin x - \ldots,$$

$$\frac{dp}{dy} = \frac{d\alpha}{dt} e^{-y} \sin x - \frac{d\alpha'}{dt} e^{-y} \cos x + \ldots + g + e^{-2y} (\alpha^2 + \alpha'^2).$$

In forming equation (6) to the second order of small quantities we need to include only the principal term of u, but v must be taken correct to the second order. As the equation of the free surface we assume

$$y = a \cos x + a' \sin x + b \cos 2x + b' \sin 2x + c \cos 3x + c' \sin 3x + \ldots \quad \ldots(7)$$

in which b, b', c, c', are small compared with a, a'. Thus (6) gives

$$(1 - a\cos x - a'\sin x)\left(-\frac{d^2\alpha}{dt^2}\sin x + \frac{d^2\alpha'}{dt^2}\cos x\right) - \frac{d^2\beta}{dt^2}\sin 2x$$

$$+\frac{d^2\beta'}{dt^2}\cos 2x - \frac{d^2\gamma}{dt^2}\sin 3x + \frac{d^2\gamma}{dt^2}\cos 3x - \alpha\frac{d\alpha}{dt} - \alpha'\frac{d\alpha'}{dt} + F'$$

$$-(\alpha\cos x + \alpha'\sin x)\left(\frac{d\alpha}{dt}\cos x + \frac{d\alpha'}{dt}\sin x\right) - \{(1 - a\cos x - a'\sin x)$$

$$\times(\alpha\sin x - \alpha'\cos x) + 2\beta\sin 2x - 2\beta'\cos 2x + 3\gamma\sin 3x - 3\gamma'\cos 3x\}$$

$$\times\left\{g + \frac{d\alpha}{dt}\sin x - \frac{d\alpha'}{dt}\cos x\right\} = 0. \quad\text{.......................}(8)$$

This equation is to hold good to the second order for all values of x, and therefore for each Fourier component separately. The terms in $\sin x$ and $\cos x$ give

$$\frac{d^2\alpha}{dt^2} + g\alpha = 0, \qquad \frac{d^2\alpha'}{dt^2} + g\alpha' = 0. \quad\text{...................}(9)$$

The term in $\sin 2x$ gives

$$\frac{d^2\beta}{dt^2} + 2g\beta = \frac{a}{2}\left(\frac{d^2\alpha}{dt^2} + g\alpha\right) - \frac{a'}{2}\left(\frac{d^2\alpha'}{dt^2} + g\alpha'\right) = 0, \quad\text{.........}(10)$$

and, similarly, that in $\cos 2x$ gives

$$\frac{d^2\beta'}{dt^2} + 2g\beta'' = 0. \quad\text{.........................}(11)$$

In like manner

$$\frac{d^2\gamma}{dt^2} + 3g\gamma = 0, \qquad \frac{d^2\gamma'}{dt^2} + 3g\gamma' = 0,\text{.................}(12)$$

and so on. These are the results of the surface condition $Dp/Dt = 0$. From the other surface condition ($p = 0$) we find in the same way

$$-\frac{d\alpha}{dt} + ga' = 0, \qquad \frac{d\alpha'}{dt} + ga = 0. \quad\text{.................}(13)$$

$$gb' = \frac{d\beta}{dt} + \frac{a'}{2}\frac{d\alpha'}{dt} - \frac{a}{2}\frac{d\alpha}{dt} = \frac{d\beta}{dt} - gaa'. \quad\text{..............}(14)$$

$$gb = -\frac{d\beta}{dt} + \frac{a'}{2}\frac{d\alpha}{dt} + \frac{a}{2}\frac{d\alpha'}{dt} = -\frac{d\beta}{dt} + \tfrac{1}{2}g\,(a'^2 - a^2). \quad\text{.........}(15)$$

$$-\frac{d\gamma}{dt} + gc' = 0, \qquad \frac{d\gamma'}{dt} + gc = 0. \quad\text{....................}(16)$$

From equations (9) to (16) we see that a, a' satisfy the same equations (9) as do α, α', and also that c, c' satisfy the same equations (12) as do γ, γ'; but that b, b' are not quite so simply related to β, β'.

Let us now suppose that the principal terms represent a progressive wave. In accordance with (9) we may take

$$a = A\cos t', \qquad a' = A\sin t', \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

where $t' = \sqrt{g}\,.\,t$. Then if β, β', γ, γ', do not appear, c, c', are zero, and $b = \frac{1}{2}A^2(\sin^2 t' - \cos^2 t')$, $b' = -A^2\cos t'\sin t'$; so that

$$y = A\cos(x - t') - \tfrac{1}{2}A^2\cos 2(x - t'), \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

representing a permanent wave-form propagated with velocity $\sqrt{g}$. So far as it goes, this agrees with (1). But now in addition to these terms we may have others, for which b, b' need only to satisfy

$$(d^2/dt'^2 + 2)(b, b') = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

and c, c' need only to satisfy

$$(d^2/dt'^2 + 3)(c, c') = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

The corresponding terms in y represent merely such waves, propagated in either direction, and of wave-lengths equal to an aliquot part of the principal wave-length, as might exist alone of infinitesimal height, when there is no primary wave at all. When these are included, the aggregate, even though it be all propagated in the same direction, loses its character of possessing a permanent wave-shape, and further it has no tendency to acquire such a character as time advances.

If the principal wave is *stationary* we may take

$$a = A\cos t', \qquad a' = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

If β, β', γ, γ', vanish,

$$b = -\tfrac{1}{2}a^2, \quad b' = 0, \quad c = 0, \quad c' = 0,$$

and

$$y = A\cos x\,.\cos t' - \tfrac{1}{2}A^2\cos 2x\,.\cos^2 t'. \ldots\ldots\ldots\ldots\ldots (22)$$

According to (22) the surface comes to its zero position everywhere when $\cos t' = 0$, and the displacement is a maximum when $\cos t' = \pm 1$. Then

$$y = \pm A\cos x - \tfrac{1}{2}A^2\cos 2x, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

so that at this moment the wave-form is the same as for the progressive wave (18). Since y is measured downwards, the maximum elevation above the mean level exceeds numerically the maximum depression below it.

In the more general case (still with β, etc., evanescent) we may write

$$a = A\cos t' + B\sin t', \quad a' = A'\cos t' + B'\sin t',$$

with

$$b' = -aa', \quad b = \tfrac{1}{2}(a'^2 - a^2), \quad c' = 0, \quad c = 0.$$

When β, β', γ, γ', are finite, waves such as might exist alone, of lengths equal to aliquot parts of the principal wave-length and of corresponding frequencies, are superposed. In these waves the amplitude and phase are arbitrary.

When we retain the third order of small quantities, the equations naturally become more complicated. We now assume that in (3) β, β', are small quantities of the second order, and γ, γ', small quantities of the third order. For p, as an extension of (5), we get

$$p = e^{-y}\left(-\frac{d\alpha}{dt}\sin x + \frac{d\alpha'}{dt}\cos x\right) + e^{-2y}\left(-\frac{d\beta}{dt}\sin 2x + \frac{d\beta'}{dt}\cos 2x\right)$$
$$+ e^{-3y}\left(-\frac{d\gamma}{dt}\sin 3x + \frac{d\gamma'}{dt}\cos 3x\right) + gy + F - \tfrac{1}{2}e^{-2y}(\alpha^2 + \alpha'^2)$$
$$- 2e^{-3y}\{(\alpha\beta + \alpha'\beta')\cos x + (\alpha\beta' - \alpha'\beta)\sin x\}. \quad \text{.................(24)}$$

This is to be made to vanish at the surface. Also we find, on reduction,

$$-\frac{Dp}{Dt} = (1 - y + \tfrac{1}{2}y^2)\left\{\left(\frac{d^2\alpha}{dt^2} + g\alpha\right)\sin x - \left(\frac{d^2\alpha'}{dt^2} + g\alpha'\right)\cos x\right\}$$
$$+ (1 - 2y)\left\{\left(\frac{d^2\beta}{dt^2} + 2g\beta\right)\sin 2x - \left(\frac{d^2\beta'}{dt^2} + 2g\beta'\right)\cos 2x\right\}$$
$$+ \left(\frac{d^2\gamma}{dt^2} + 3g\gamma\right)\sin 3x - \left(\frac{d^2\gamma'}{dt^2} + 3g\gamma'\right)\cos 3x - F'$$
$$+ 2(1 - 2y)\left(\alpha\frac{d\alpha}{dt} + \alpha'\frac{d\alpha'}{dt}\right) + 4\sin x\frac{d}{dt}(\alpha\beta' - \alpha'\beta)$$
$$+ 4\cos x\frac{d}{dt}(\alpha\beta' + \alpha'\beta) + (\alpha^2 + \alpha'^2)(\alpha\sin x - \alpha'\cos x); \quad \text{......(25)}$$

and at the surface $Dp/Dt = 0$ for all values of x. In (25) y is of the form (7), where b, b', are of the second order, c, c', of the third order.

Considering the coefficients of $\sin x$, $\cos x$, in (25) when reduced to Fourier's form, we see that $d^2\alpha/dt^2 + g\alpha$, $d^2\alpha'/dt^2 + g\alpha'$, are both of the third order of small quantities, so that in the first line the factor $(1 - y + \frac{1}{2}y^2)$ may be replaced by unity. Again, from the coefficients of $\sin 2x$, $\cos 2x$, we see that to the third order inclusive

$$\frac{d^2\beta}{dt^2} + 2g\beta = 0, \qquad \frac{d^2\beta'}{dt^2} + 2g\beta' = 0, \quad \text{.................(26)}$$

and from the coefficients of $\sin 3x$, $\cos 3x$ that to the third order inclusive

$$\frac{d^2\gamma}{dt^2} + 3g\gamma = 0, \qquad \frac{d^2\gamma'}{dt^2} + 3g\gamma' = 0. \quad \text{.................(27)}$$

And now returning to the coefficients of $\sin x$, $\cos x$, we get

$$\frac{d^2\alpha}{dt^2} + g\alpha - 2\alpha'\frac{d}{dt}(\alpha^2 + \alpha'^2) + 4\frac{d}{dt}(\alpha\beta' - \alpha'\beta) + \alpha(\alpha^2 + \alpha'^2) = 0, \quad \text{...(28)}$$

$$\frac{d^2\alpha'}{dt^2} + g\alpha' + 2\alpha\frac{d}{dt}(\alpha^2 + \alpha'^2) - 4\frac{d}{dt}(\alpha\beta' + \alpha'\beta) + \alpha'(\alpha^2 + \alpha'^2) = 0. \quad (29)$$

Passing next to the condition $p=0$, we see from (24), by considering the coefficients of $\sin x$, $\cos x$, that

$$-\frac{d\alpha}{dt}+g a'+\text{terms of 3rd order}=0,$$

$$\frac{d\alpha'}{dt}+g a+\text{terms of 3rd order}=0.$$

The coefficients of $\sin 2x$, $\cos 2x$, require, as in (14), (15), that

$$b'=\frac{1}{g}\frac{d\beta}{dt}-aa', \qquad b=-\frac{1}{g}\frac{d\beta'}{dt}+\frac{a'^2-a^2}{2}. \quad\text{..............(30)}$$

Again, the coefficients of $\sin 3x$, $\cos 3x$, give

$$c'=\frac{1}{g}\frac{d\gamma}{dt}-\tfrac{3}{2}(a'b+ab')+\tfrac{3}{8}a'(a'^2-3a^2)$$

$$=\frac{1}{g}\left\{\frac{d\gamma}{dt}-\frac{3a}{2}\frac{d\beta}{dt}+\frac{3a'}{2}\frac{d\beta'}{dt}\right\}-\frac{3a'}{8}(a'^2-3a^2), \quad\text{...........(31)}$$

$$c=-\frac{1}{g}\frac{d\gamma'}{dt}+\tfrac{3}{2}(a'b'-ab)+\frac{3a}{8}(3a'^2-a^2)$$

$$=\frac{1}{g}\left\{-\frac{d\gamma'}{dt}+\frac{3a'}{2}\frac{d\beta}{dt}+\frac{3a}{2}\frac{d\beta'}{dt}\right\}-\frac{3a}{8}(3a'^2-a^2). \quad\text{.........(32)}$$

When β, β', γ, γ', vanish, these results are much simplified. We have

$$b'=-aa', \qquad b=\tfrac{1}{2}(a'^2-a^2), \quad\text{.....................(33)}$$

$$c'=-\frac{3a'}{8}(a'^2-3a^2), \qquad c=-\frac{3a}{8}(3a'^2-a^2). \quad\text{...........(34)}$$

If the principal terms represent a purely progressive wave, we may take, as in (17),

$$a=A\cos nt, \qquad a'=A\sin nt, \quad\text{.....................(35)}$$

where n is for the moment undetermined. Accordingly

$$b'=-\tfrac{1}{2}A^2\sin 2nt, \qquad b=-\tfrac{1}{2}A^2\cos 2nt,$$

$$c'=\tfrac{3}{8}A^3\sin 3nt, \qquad c=\tfrac{3}{8}A^3\cos 3nt;$$

so that

$$y=A\cos(x-nt)-\tfrac{1}{2}A^2\cos 2(x-nt)+\tfrac{3}{8}A^3\cos 3(x-nt), \quad\text{.......(36)}$$

representing a progressive wave of permanent type, as found by Stokes.

To determine n we utilize (28), (29), in the small terms of which we may take

$$\alpha=g\int a'dt=-\frac{gA}{n}\cos nt, \qquad \alpha'=-g\int a\,dt=-\frac{gA}{n}\sin nt,$$

so that

$$\alpha^2+\alpha'^2=g^2A^2/n^2.$$

Thus

$$\frac{d^2(\alpha,\alpha')}{dt^2}+\left(g+\frac{g^2A^2}{n^2}\right)(\alpha,\alpha')=0,$$

and

$$n^2=g+g^2A^2/n^2=g(1+A^2), \quad\text{....................(37)}$$

or, if we restore homogeneity by introduction of $k\,(=2\pi/\lambda)$,

$$n^2 = g/k\,.\,(1 + k^2A^2). \qquad\qquad (38)$$

Let us next suppose that the principal terms represent a stationary, instead of a progressive, wave and take

$$a = A\cos nt, \qquad a' = 0. \qquad\qquad (39)$$

Then by (33), (34),

$$b' = 0, \qquad b = -\tfrac{1}{2}A^2\cos^2 nt, \qquad c' = 0, \qquad c = \tfrac{3}{8}A^3\cos^3 nt;$$

and

$$y = A\cos nt\cos x - \tfrac{1}{2}A^2\cos^2 nt\cos 2x + \tfrac{3}{8}A^3\cos^3 nt\cos 3x. \qquad (40)$$

When $\cos nt = 0$, $y = 0$ throughout; when $\cos nt = 1$,

$$y = A\cos x - \tfrac{1}{2}A^2\cos 2x + \tfrac{3}{8}A^3\cos 3x,$$

so that at this moment of maximum displacement the form is the same as for the progressive wave (36).

We have still to determine n so as to satisfy (28), (29), with evanescent β, β'. The first is satisfied by $\alpha = 0$, since $a' = 0$. The second becomes

$$\frac{d^2\alpha'}{dt^2} + g\alpha' + 4a\alpha'\frac{d\alpha'}{dt} + \alpha'^3 = 0.$$

In the small terms we may take $\alpha' = -g\int a\,dt = -\frac{gA}{n}\sin nt$, so that

$$\frac{d^2\alpha'}{dt^2} + g\alpha' + \frac{g^2A^3}{4n}(\sin nt + 5\sin 3nt) = 0.$$

To satisfy this we assume

$$\alpha' = H\sin nt + K\sin 3nt.$$

Then $$H(g - n^2) + \frac{g^2A^3}{4n} = 0, \qquad K(g - 9n^2) + \frac{5g^2A^3}{4n} = 0,$$

from the first of which

$$n^2 = g + \frac{g^2A^3}{4nH} = g - \frac{gA^2}{4}, \qquad\qquad (41)$$

or, if we restore homogeneity by introduction of k,

$$n^2 = g/k\,.\,(1 - \tfrac{1}{4}k^2A^2). \qquad\qquad (42)$$

With this value of n the stationary vibration

$$y = A\cos nt\cos kx - \tfrac{1}{2}kA^2\cos^2 nt\cos 2kx + \tfrac{3}{8}k^2A^3\cos^3 nt\cos 3kx, \quad (43)$$

satisfies all the conditions. It may be remarked that according to (42) the frequency of vibration is diminished by increase of amplitude.

The special cases above considered of purely progressive or purely stationary waves possess an exceptional simplicity. In general, with omission of β, β', equations (28), (29), become

$$\frac{d^2\alpha}{dt^2} + g\alpha - \frac{2}{g}\frac{d\alpha}{dt}\frac{d(\alpha^2 + \alpha'^2)}{dt} + \alpha(\alpha^2 + \alpha'^2) = 0, \qquad (44)$$

and a like equation in which α and α' are interchanged. In the terms of the third order, we take

$$\alpha = P\cos nt + Q\sin nt, \qquad \alpha' = P'\cos nt + Q'\sin nt, \qquad \text{.........(45)}$$

so that

$$\alpha^2 + \alpha'^2 = \tfrac{1}{2}(P^2 + Q^2 + P'^2 + Q'^2) + \tfrac{1}{2}(P^2 + P'^2 - Q^2 - Q'^2)\cos 2nt$$
$$+ (PQ + P'Q')\sin 2nt.$$

The third order terms in (44) are

$$\tfrac{1}{2}(P^2 + P'^2 + Q^2 + Q'^2)(P\cos nt + Q\sin nt)$$
$$+ 2\cos nt\cos 2nt\left\{\tfrac{1}{4}P(P^2 + P'^2 - Q^2 - Q'^2) - \frac{2n^2Q}{g}(PQ + P'Q')\right\}$$
$$+ 2\sin nt\sin 2nt\left\{\tfrac{1}{2}Q(PQ + P'Q') - \frac{n^2P}{g}(P^2 + P'^2 - Q^2 - Q'^2)\right\}$$
$$+ 2\sin nt\cos 2nt\left\{\tfrac{1}{4}Q(P^2 + P'^2 - Q^2 - Q'^2) + \frac{2n^2P}{g}(PQ + P'Q')\right\}$$
$$+ 2\cos nt\sin 2nt\left\{\tfrac{1}{2}P(PQ + P'Q') + \frac{n^2Q}{g}(P^2 + P'^2 - Q^2 - Q'^2)\right\},$$

of which the part in $\sin nt$ has the coefficient

$$Q\{\tfrac{1}{4}(P^2 + P'^2) + \tfrac{3}{4}(Q^2 + Q'^2)\} + \tfrac{1}{2}P(PQ + P'Q')$$
$$+ n^2/g\,.\{Q(P^2 + P'^2 - Q^2 - Q'^2) - 2P(PQ + P'Q')\}$$

or, since $n^2 = g$ approximately,

$$Q\{\tfrac{5}{4}(P^2 + P'^2) - \tfrac{1}{4}(Q^2 + Q'^2)\} - \tfrac{3}{2}P(PQ + P'Q'). \qquad \text{.........(46)}$$

In like manner the coefficient of $\cos nt$ is

$$P\{\tfrac{5}{4}(Q^2 + Q'^2) - \tfrac{1}{4}(P^2 + P'^2)\} - \tfrac{3}{2}Q(PQ + P'Q'), \qquad \text{.........(47)}$$

differing merely by the interchange of P and Q.

But when these values are employed in (44), it is not, in general, possible, with constant values of P, Q, P', Q', to annul the terms in $\sin nt$, $\cos nt$. We obtain from the first

$$n^2 = g + \tfrac{5}{4}(P^2 + P'^2) - \tfrac{1}{4}(Q^2 + Q'^2) - \frac{3P}{2Q}(PQ + P'Q'), \qquad \text{......(48)}$$

and from the second

$$n^2 = g + \tfrac{5}{4}(Q^2 + Q'^2) - \tfrac{1}{4}(P^2 + P'^2) - \frac{3Q}{2P}(PQ + P'Q'); \qquad \text{......(49)}$$

and these are inconsistent, unless

$$(PP' + QQ')(PQ' - P'Q) = 0. \qquad \text{....................(50)}$$

The latter condition is unaltered by interchange of dashed and undashed letters, and thus it serves equally for the equation in α'.

The two alternatives indicated in (50) correspond to the particular cases already considered. In the first ($PP' + QQ' = 0$) we have a purely progressive wave and in the second a purely stationary one.

When the condition (50) does not hold good, it is impossible to satisfy our equations as before with constant values of n, P, Q, P', Q'; and it is perhaps hardly worth while to pursue the more complicated questions which then arise. It may suffice to remark that an approximately stationary wave can never pass into an approximately progressive wave, nor *vice versâ*. The progressive wave has momentum, while the stationary wave has none, and momentum is necessarily conserved.

When β, β', γ, γ', are not zero, additional terms enter. Equations (26), (30), show that the additions to b, b', vary as the sine and cosine of $\sqrt{(2g)} \,.\, t$, and represent waves which might exist in the complete absence of the principal wave.

The additions to c, c', are more complicated. As regards the parts depending in (31), (32), on $d\gamma/dt$, $d\gamma'/dt$, they are proportional to the sine and cosine of $\sqrt{(3g)} \,.\, t$, and represent waves which might exist alone. But besides these there are other parts, analogous to the combination-tones of Acoustics, resulting from the interaction of the β-waves with the principal wave. These vary as the sine and cosine of $\sqrt{g} \,.\, \{\sqrt{2} \pm 1\}\, t$, thus possessing frequencies differing from the former frequencies. Similar terms will enter into the expression for n^2 as determined from (28), (29).

In the particular case of β, β', vanishing, even though γ, γ' (assumed still to be of the third order) remain, we recover most of the former simplicity, the only difference being the occurrence in c, c', of terms in $\sqrt{(3g)} \,.\, t$, such as might exist alone.

ÆOLIAN TONES.

[*Philosophical Magazine*, Vol. XXIX. pp. 433—444, 195, 1915.]

IN what has long been known as the Æolian Harp, a stretched string, such as a pianoforte wire or a violin string, is caused to vibrate in one of its possible modes by the impact of wind; and it was usually supposed that the action was analogous to that of a violin bow, so that the vibrations were executed in the plane containing the direction of the wind. A closer examination showed, however, that this opinion was erroneous and that in fact the vibrations are transverse to the wind*. It is not essential to the production of sound that the string should take part in the vibration, and the general phenomenon, exemplified in the whistling of wind among trees, has been investigated by Strouhal† under the name of *Reibungstöne*.

In Strouhal's experiments a vertical wire or rod attached to a suitable frame was caused to revolve with uniform velocity about a parallel axis. The pitch of the æolian tone generated by the relative motion of the wire and of the air was found to be independent of the length and of the tension of the wire, but to vary with the diameter (D) and with the speed (V) of the motion. Within certain limits the relation between the frequency of vibration (N) and these data was expressible by

$$N = \cdot 185\, V/D, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)‡$$

the centimetre and the second being units.

When the speed is such that the æolian tone coincides with one of the proper tones of the wire, supported so as to be capable of free independent vibration, the sound is greatly reinforced, and with this advantage Strouhal found it possible to extend the range of his observations. Under the more extreme conditions then practicable the observed pitch deviated considerably

* *Phil. Mag.* Vol. VII. p. 149 (1879); *Scientific Papers*, Vol. I. p. 413.

† *Wied. Ann.* Vol. V. p. 216 (1878).

‡ In (1) V is the velocity of the wire relatively to the walls of the laboratory.

from the value given by (1). He further showed that with a given diameter and a given speed a rise of temperature was attended by a fall in pitch.

If, as appears probable, the compressibility of the fluid may be left out of account, we may regard N as a function of the relative velocity V, D, and ν the kinematic coefficient of viscosity. In this case N is necessarily of the form

$$N = V/D \,.\, f(\nu/VD), \quad \text{.........................(2)}$$

where f represents an arbitrary function; and there is dynamical similarity, if $\nu \propto VD$. In observations upon air at one temperature ν is constant; and if D vary inversely as V, ND/V should be constant, a result fairly in harmony with the observations of Strouhal. Again, if the temperature rises, ν increases, and in order to accord with observation, we must suppose that the function f diminishes with increasing argument.

"An examination of the actual values in Strouhal's experiments shows that ν/VD was always small; and we are thus led to represent f by a few terms of MacLaurin's series. If we take

$$f(x) = a + bx + cx^2,$$

we get

$$N = a\frac{V}{D} + b\frac{\nu}{D^2} + c\frac{\nu^2}{VD^3}. \quad \text{.........................(3)}$$

"If the third term in (3) may be neglected, the relation between N and V is linear. This law was formulated by Strouhal, and his diagrams show that the coefficient b is negative, as is also required to express the observed effect of a rise of temperature. Further,

$$D\frac{dN}{dV} = a - \frac{c\nu^2}{V^2D^2}, \quad \text{.........................(4)}$$

so that $D\,dN/dV$ is very nearly constant, a result also given by Strouhal on the basis of his measurements.

"On the whole it would appear that the phenomena are satisfactorily represented by (2) or (3), but a dynamical theory has yet to be given. It would be of interest to extend the experiments to liquids*."

Before the above paragraphs were written I had commenced a systematic deduction of the form of f from Strouhal's observations by plotting ND/V against VD. Lately I have returned to the subject, and I find that nearly all his results are fairly well represented by two terms of (3). In C.G.S. measure

$$\frac{ND}{V} = \cdot195\left(1 - \frac{3\cdot02}{VD}\right) \text{.........................(5)}$$

Although the agreement is fairly good, there are signs that a change of wire introduces greater discrepancies than a change in V—a circumstance

* *Theory of Sound*, 2nd ed. Vol. II. § 372 (1896).

which may possibly be attributed to alterations in the character of the surface. The simple form (2) assumes that the wires are smooth, or else that the roughnesses are in proportion to D, so as to secure geometrical similarity.

The completion of (5) from the theoretical point of view requires the introduction of ν. The temperature for the experiments in which ν would enter most was about 20° C., and for this temperature

$$\nu = \frac{\mu}{\rho} = \frac{1806 \times 10^{-7}}{\cdot 00120} = \cdot 1505 \text{ C.G.S.}$$

The generalized form of (5) is accordingly

$$\frac{ND}{V} = \cdot 195 \left(1 - \frac{20\cdot 1\,\nu}{VD}\right), \quad \text{.........................(6)}$$

applicable now to any fluid when the appropriate value of ν is introduced. For water at 15° C., $\nu = \cdot 0115$, much *less* than for air.

Strouhal's observations have recently been discussed by Krüger and Lauth*, who appear not to be acquainted with my theory. Although they do not introduce viscosity, they recognize that there is probably some cause for the observed deviations from the simplest formula (1), other than the complication arising from the circulation of the air set in motion by the revolving parts of the apparatus. Undoubtedly this circulation marks a weak place in the method, and it is one not easy to deal with. On this account the numerical quantities in (6) may probably require some correction in order to express the true formula when V denotes the velocity of the wire through otherwise undisturbed fluid.

We may find confirmation of the view that viscosity enters into the question, much as in (6), from some observations of Strouhal on the effect of *temperature*. Changes in ν will tell most when VD is small, and therefore I take Strouhal's table XX., where $D = \cdot 0179$ cm. In this there appears

$$t_1 = 11°, \quad V_1 = 385, \quad N_1/V_1 = 6\cdot 70, \quad \nu_1,$$

$$t_2 = 31°, \quad V_2 = 381, \quad N_2/V_2 = 6\cdot 48, \quad \nu_2.$$

Introducing these into (6), we get

$$6\cdot 70 - 6\cdot 48 = \frac{\cdot 195}{D}\left(1 - \frac{20\cdot 1\,\nu_1}{V_1 D}\right) - \frac{\cdot 195}{D}\left(1 - \frac{20\cdot 1\,\nu_2}{V_2 D}\right),$$

or with sufficient approximation

$$\nu_2 - \nu_1 = \frac{\cdot 52\, D^2 V}{\cdot 195 \times 20\cdot 1} = \cdot 016 \text{ C.G.S.}$$

* "Theorie der Hiebtöne," *Ann. d. Physik*, Vol. XLIV. p. 801 (1914).

We may now compare this with the known values of ν for the temperatures in question. We have

$$\mu_{31} = 1853 \times 10^{-7}, \quad \rho_{31} = \cdot 001161,$$

$$\mu_{11} = 1765 \times 10^{-7}, \quad \rho_{11} = \cdot 001243;$$

so that $$\nu_2 = \cdot 1596, \qquad \nu_1 = \cdot 1420,$$

and $$\nu_2 - \nu_1 = \cdot 018.$$

The difference in the values of ν at the two temperatures thus accounts in (6) for the change of frequency both in sign and in order of magnitude.

As regards dynamical explanation it was evident all along that the origin of vibration was connected with the instability of the vortex sheets which tend to form on the two sides of the obstacle, and that, at any rate when a wire is maintained in transverse vibration, the phenomenon must be unsymmetrical. The alternate formation in water of detached vortices on the two sides is clearly described by H. Bénard*. "Pour une vitesse suffisante, au-dessous de laquelle il n'y a pas de tourbillons (cette vitesse limite croît avec la viscosité et decroît quand l'épaisseur transversale des obstacles augmente), *les tourbillons produits périodiquement se détachent alternativement à droite et à gauche du remous d'arrière qui suit le solide; ils gagnent presque immédiatement leur emplacement définitif*, *de sorte qu'à l'arrière de l'obstacle se forme une double rangée alternée d'entonnoirs stationnaires, ceux de droite dextrogyres, ceux de gauche lévogyres, séparés par des intervalles égaux.*"

The symmetrical and unsymmetrical processions of vortices were also figured by Mallock† from direct observation.

In a remarkable theoretical investigation‡ Kármán has examined the question of the stability of such processions. The fluid is supposed to be incompressible, to be devoid of viscosity, and to move in two dimensions. The vortices are concentrated in points and are disposed at equal intervals (l) along two parallel lines distant h. Numerically the vortices are all equal, but those on different lines have opposite signs.

Apart from stability, steady motion is possible in two arrangements (a) and (b), fig. 1, of which (a) is symmetrical. Kármán shows that (a) is always unstable, whatever may be the ratio of h to l; and further that (b) is usually unstable also. The single exception occurs when $\cosh(\pi h/l) = \sqrt{2}$, or $h/l = 0{\cdot}283$. With this ratio of h/l, (b) is stable for every kind of displacement except one, for which there is neutrality. The only procession which can possess a practical permanence is thus defined.

* *C. R.* t. 147, p. 839 (1908).

† *Proc. Roy. Soc.* Vol. LXXXIV. A, p. 490 (1910).

‡ *Göttingen Nachrichten*, 1912, Heft 5, S. 547; Kármán and Rubach, *Physik. Zeitschrift*, 1912, p. 49. I have verified the more important results.

The corresponding motion is expressed by the complex potential (ϕ potential, ψ stream-function)

$$\phi + i\psi = \frac{i\zeta}{2\pi} \log \frac{\sin\{\pi(z_0 - z)/l\}}{\sin\{\pi(z_0 + z)/l\}}, \quad \text{.................(7)}$$

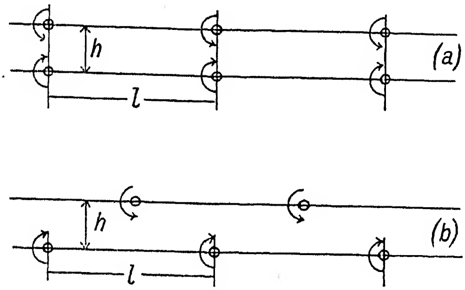

Fig. 1.

in which ζ denotes the strength of a vortex, $z = x + iy$, $z_0 = \frac{1}{4}l + ih$. The x-axis is drawn midway between the two lines of vortices and the y-axis halves the distance between neighbouring vortices with opposite rotation. Kármán gives a drawing of the stream-lines thus defined.

The constant velocity of the processions is given by

$$u = \frac{\zeta}{2l} \tanh \frac{\pi h}{l} = \frac{\zeta}{l\sqrt{8}}. \quad \text{.......................(8)}$$

This velocity is relative to the fluid at a distance.

The observers who have experimented upon water seem all to have used obstacles not susceptible of vibration. For many years I have had it in my mind to repeat the æolian harp effect with water*, but only recently have brought the matter to a test. The water was contained in a basin, about 36 cm. in diameter, which stood upon a sort of turn-table. The upper part, however, was not properly a table, but was formed of two horizontal beams crossing one another at right angles, so that the whole apparatus resembled rather a turn-*stile*, with four spokes. It had been intended to drive from a small water-engine, but ultimately it was found that all that was needed could more conveniently be done by hand after a little practice. A metronome beat approximate half seconds, and the spokes (which projected beyond the basin) were pushed gently by one or both hands until the rotation was uniform with passage of one or two spokes in correspondence with an assigned number of beats. It was necessary to allow several minutes in order to

* From an old note-book. "Bath, Jan. 1884. I find in the baths here that if the spread fingers be drawn pretty quickly through the water (palm foremost was best), they are thrown into transverse vibration and strike one another. This seems like æolian string.... The blade of a flesh-brush about $1\frac{1}{2}$ inch broad seemed to vibrate transversely in its own plane when moved through water broadways forward. It is pretty certain that with proper apparatus these vibrations might be developed and observed."

make sure that the water had attained its ultimate velocity. The axis of rotation was indicated by a pointer affixed to a small stand resting on the bottom of the basin and rising slightly above the level of the water.

The pendulum (fig. 2), of which the lower part was immersed, was supported on two points (A, B) so that the possible vibrations were limited to one vertical plane. In the usual arrangement the vibrations of the rod would be radial, *i.e.* transverse to the motion of the water, but it was easy to turn the pendulum round when it was desired to test whether a circumferential vibration could be maintained. The rod C itself was of brass tube $8\frac{1}{2}$ mm. in diameter, and to it was clamped a hollow cylinder of lead D. The time

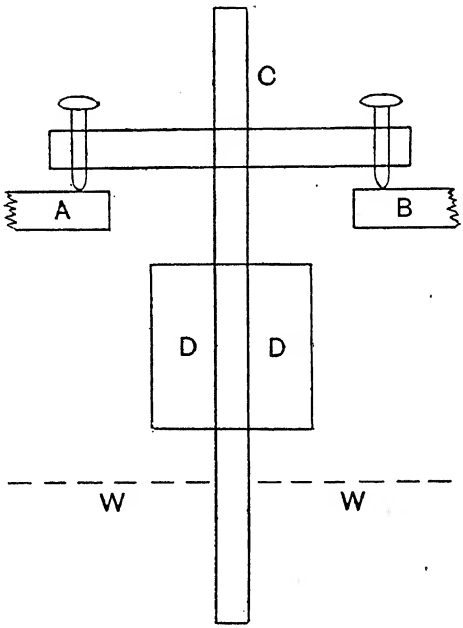

Fig. 2.

of complete vibration (τ) was about half a second. When it was desired to change the diameter of the immersed part, the rod C was drawn up higher and prolonged below by an additional piece—a change which did not much affect the period τ. In all cases the length of the part immersed was about 6 cm.

Preliminary observations showed that in no case were vibrations generated when the pendulum was so mounted that the motion of the rod would be circumferential, viz. in the direction of the stream, agreeably to what had been found for the æolian harp. In what follows the vibrations, if any, are radial, that is transverse to the stream.

In conducting a set of observations it was found convenient to begin with the highest speed, passing after a sufficient time to the next lower, and so on,

with the minimum of intermission. I will take an example relating to the main rod, whose diameter (D) is $8\frac{1}{2}$ mm., $\tau = 60/106$ sec., beats of metronome 62 in 30 sec. The speed is recorded by the number of beats corresponding to the passage of *two* spokes, and the vibration of the pendulum (after the lapse of a sufficient time) is described as small, fair, good, and so on. Thus on Dec. 21, 1914:

2 spokes to 4 beats gave fair vibration,
. . . . 5 . . . . good . . . . . .
. . . . 6 . . . . . rather more . . .
. . . . 7 . . . . good . . . .
. . . . 8 . . . . fair . . . . . .

from which we may conclude that the maximum effect corresponds to 6 beats, or to a time (T) of revolution of the turn-table equal to $2 \times 6 \times 30/62$ sec. The distance (r) of the rod from the axis of rotation was 116 mm., and the speed of the water, supposed to move with the basin, is $2\pi r/T$. The result of the observations may intelligibly be expressed by the ratio of the distance travelled by the water during one complete vibration of the pendulum to the diameter of the latter, viz.

$$\frac{\tau \,.\, 2\pi r/T}{D} = \frac{2\pi \times 116 \times 62}{8{\cdot}5 \times 6 \times 106} = 8{\cdot}36.$$

Concordant numbers were obtained on other occasions.

In the above calculation the speed of the water is taken as if it were rigidly connected with the basin, and must be an over estimate. When the pendulum is away, the water may be observed to move as a solid body after the rotation has been continued for two or three minutes. For this purpose the otherwise clean surface may be lightly dusted over with sulphur. But when the pendulum is immersed, the rotation is evidently hindered, and that not merely in the neighbourhood of the pendulum itself. The difficulty thence arising has already been referred to in connexion with Strouhal's experiments and it cannot easily be met in its entirety. It may be mitigated by increasing r, or by diminishing D. The latter remedy is easily applied up to a certain point, and I have experimented with rods 5 mm. and $3\frac{1}{2}$ mm. in diameter. With a 2 mm. rod no vibration could be observed. The final results were thus tabulated:

Diameter ...	8·5 mm.	5·0 mm.	3·5 mm.
Ratio ...	8·35	7·5	7·8

from which it would appear that the disturbance is not very serious. The difference between the ratios for the 5·0 mm. and 3·5 mm. rods is hardly outside the limits of error; and the prospect of reducing the ratio much below 7 seemed remote.

The instinct of an experimenter is to try to get rid of a disturbance, even though only partially; but it is often equally instructive to increase it. The

observations of Dec. 21 were made with this object in view; besides those already given they included others in which the disturbance due to the vibrating pendulum was augmented by the addition of a similar rod ($8\frac{1}{2}$ mm.) immersed to the same depth and situated symmetrically on the same diameter of the basin. The anomalous effect would thus be doubled. The record was as follows:

2 spokes to	3 beats gave	little or no vibration,	
....	4	fair	......
....	5	large	......
....	6	less	......
....	7	little or no	......

As the result of this and another day's similar observations it was concluded that the 5 beats with additional obstruction corresponded with 6 beats without it. An approximate correction for the disturbance due to improper action of the pendulum may thus be arrived at by decreasing the calculated ratio in the proportion of 6 : 5; thus

$$\tfrac{5}{6}(8{\cdot}35) = 7{\cdot}0$$

is the ratio to be expected in a uniform stream. It would seem that this cannot be far from the mark, as representing the travel at a distance from the pendulum in an otherwise uniform stream during the time of one complete vibration of the latter. Since the correction for the other diameters will be decidedly less, the above number may be considered to apply to all three diameters experimented on.

In order to compare with results obtained from air, we must know the value of ν/VD. For water at 15° C. $\nu = \mu = {\cdot}0115$ C.G.S.; and for the 8·5 mm. pendulum $\nu/VD = {\cdot}0011$. Thus from (6) it appears that ND/V should have nearly the full value, say ·190. The reciprocal of this, or 5·3, should agree with the ratio found above as 7·0; and the discrepancy is larger than it should be.

An experiment to try whether a change of viscosity had appreciable influence may be briefly mentioned. Observations were made upon water heated to about 60° C. and at 12° C. No difference of behaviour was detected. At 60° C. $\mu = {\cdot}0049$, and at 12° C. $\mu = {\cdot}0124$.

I have described the simple pendulum apparatus in some detail, as apart from any question of measurements it demonstrates easily the general principle that the vibrations are transverse to the stream, and when in good action it exhibits very well the double row of vortices as witnessed by dimples upon the surface of the water.

The discrepancy found between the number from water (7·0) and that derived from Strouhal's experiments on air (5·3) raises the question whether

the latter can be in error. So far as I know, Strouhal's work has not been repeated; but the error most to be feared, that arising from the circulation of the air, acts in the wrong direction. In the hope of further light I have remounted my apparatus of 1879. The draught is obtained from a chimney. A structure of wood and paper is fitted to the fire-place, which may prevent all access of air to the chimney except through an elongated horizontal aperture in the front (vertical) wall. The length of the aperture is 26 inches (66 cm.), and the width 4 inches (10·2 cm.); and along its middle a gut string is stretched over bridges.

The draught is regulated mainly by the amount of fire. It is well to have a margin, as it is easy to shunt a part through an aperture at the top of the enclosure, which can be closed partially or almost wholly by a superposed card. An adjustment can sometimes be got by opening a door or window. A piece of paper thrown on the fire increases the draught considerably for about half a minute.

The string employed had a diameter of ·95 mm., and it could readily be made to vibrate (in 3 segments) in unison with a fork of pitch 256. The octave, not difficult to mistake, was verified by a resonator brought up close to the string. That the vibration is transverse to the wind is confirmed by the behaviour of the resonator, which goes out of action when held symmetrically. The sound, as heard in the open without assistance, was usually feeble, but became loud when the ear was held close to the wooden frame. The difficulty of the experiment is to determine the velocity of the wind, where it acts upon the string. I have attempted to do this by a pendulum arrangement designed to determine the wind by its action upon an elongated piece of mirror (10·1 cm. × 1·6 cm.) held perpendicularly and just in front of the string. The pendulum is supported on two points—in this respect like the one used for the water experiments; the mirror is above, and there is a counter-weight below. An arm projects horizontally forward on which a rider can be placed. In commencing observations the wind is cut off by a large card inserted across the aperture and just behind the string. The pendulum then assumes a sighted position, determined in the usual way by reflexion. When the wind operates the mirror is carried with it, but is brought back to the sighted position by use of a rider of mass equal to ·485 gm.

Observations have been taken on several occasions, but it will suffice to record one set whose result is about equal to the average. The (horizontal) distance of the rider from the axis of rotation was 62 mm., and the vertical distance of the centre line of the mirror from the same axis is 77 mm. The force of the wind upon the mirror was thus 62 × ·485 ÷ 77 gms. weight. The mean pressure P is

$$\frac{62 \times \cdot 485 \times 981}{77 \times 16\cdot 2} = 23\cdot 7 \, \frac{\text{dynes}}{\text{cm.}^2}.$$

The formula connecting the velocity of the wind V with the pressure P may be written

$$P = C\rho V^2,$$

where ρ is the density; but there is some uncertainty as to the constancy of C. It appears that for large plates $C = \cdot62$, but for a plate 2 inches square Stanton found $C = \cdot52$. Taking the latter value*, we have

$$V^2 = \frac{23\cdot7}{\cdot52\rho} = \frac{23\cdot7}{\cdot52 \times \cdot00123},$$

on introduction of the value of ρ appropriate to the circumstances of the experiment. Accordingly

$$V = 192 \text{ cm./sec.}$$

The frequency of vibration (τ^{-1}) was nearly enough 256; so that

$$\frac{V\tau}{D} = \frac{192}{256 \times \cdot095} = 7\cdot9.$$

In comparing this with Strouhal, we must introduce the appropriate value of VD, that is 19, into (5). Thus

$$\frac{V}{ND} = \frac{V\tau}{D} = 6\cdot1.$$

Whether judged from the experiments with water or from those just detailed upon air, this (Strouhal's) number would seem to be too low; but the uncertainty in the value of C above referred to precludes any very confident conclusion. It is highly desirable that Strouhal's number should be further checked by some method justifying complete confidence.

When a wire or string exposed to wind does not itself enter into vibration, the sound produced is uncertain and difficult to estimate. No doubt the wind is often different at different parts of the string, and even at the same part it may fluctuate rapidly. A remedy for the first named cause of unsteadiness is to listen through a tube, whose open end is brought pretty close to the obstacle. This method is specially advantageous if we take advantage of our knowledge respecting the mode of action, by using a tube drawn out to a narrow bore (say 1 or 2 mm.) and placed so as to face the processions of vortices behind the wire. In connexion with the fire-place arrangement the drawn out glass tube is conveniently bent round through 180° and continued to the ear by a rubber prolongation. In the wake of the obstacle the sound is well heard, even at some distance (50 mm.) behind; but little or nothing reaches the ear when the aperture is in front or at the side, even though quite close up, unless the wire is itself vibrating. But the special arrangement for

* But I confess that I feel doubts as to the diminution of C with the linear dimension. [1917. See next paper.]

a draught, where the observer is on the high pressure side, is not necessary; in a few minutes any one may prepare a little apparatus competent to show the effect. Fig. 3 almost explains itself. *A* is the drawn out glass tube

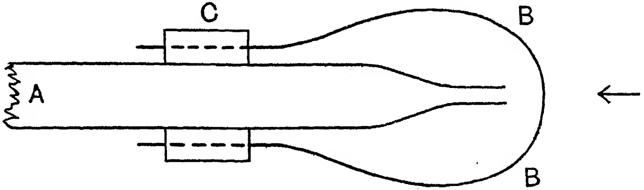

Fig. 3.

B the loop of iron or brass wire (say ·1 mm. in diameter), attached to the tube with the aid of a cork *C*. The rubber prolongation is not shown. Held in the crack of a slightly opened door or window, the arrangement yields a sound which is often pure and fairly steady.

395.

ON THE RESISTANCE EXPERIENCED BY SMALL PLATES EXPOSED TO A STREAM OF FLUID.

[*Philosophical Magazine*, Vol. xxx. pp. 179—181, 1915.]

IN a recent paper on Æolian Tones* I had occasion to determine the velocity of wind from its action upon a narrow strip of mirror (10·1 cm. × 1·6 cm.), the incidence being normal. But there was some doubt as to the coefficient to be employed in deducing the velocity from the density of the air and the force per unit area. Observations both by Eiffel and by Stanton had indicated that the resultant pressure (force reckoned per unit area) is less on small plane areas than on larger ones; and although I used provisionally a diminished value of C in the equation $P = C\rho V^2$ in view of the narrowness of the strip, it was not without hesitation†. I had in fact already commenced experiments which appeared to show that no variation in C was to be detected. Subsequently the matter was carried a little further; and I think it worth while to describe briefly the method employed. In any case I could hardly hope to attain finality, which would almost certainly require the aid of a proper wind channel, but this is now of less consequence as I learn that the matter is engaging attention at the National Physical Laboratory.

According to the principle of similitude a departure from the simple law would be most apparent when the kinematic viscosity is large and the stream velocity small. Thus, if the delicacy can be made adequate, the use of *air* resistance and such low speeds as can be reached by walking through a still atmosphere should be favourable. The principle of the method consists in balancing the two areas to be compared by mounting them upon a vertical axis, situated in their common plane, and capable of turning with the minimum of friction. If the areas are equal, their centres must be at the same distance (on opposite sides) from the axis. When the apparatus is carried forward through the air, equality of mean pressures is witnessed by the plane of the obstacles assuming a position of perpendicularity to the line of motion. If in

* *Phil. Mag.* Vol. xxix. p. 442 (1915). [Art. 394.]

† See footnote on p. [324].

this position the mean pressure on one side is somewhat deficient, the plane on that side advances against the relative stream, until a stable balance is attained in an oblique position, in virtue of the displacement (forwards) of the centres of pressure from the centres of figure.

The plates under test can be cut from thin card and of course must be accurately measured. In my experiments the axis of rotation was a sewing-needle held in a U-shaped strip of brass provided with conical indentations. The longitudinal pressure upon the needle, dependent upon the spring of the brass, should be no more than is necessary to obviate shift. The arms connecting the plates with the needle are as slender as possible consistent with the necessary rigidity, not merely in order to save weight but to minimise their resistance. They may be made of wood, provided it be accurately shaped, or of wire, preferably of aluminium. Regard must be paid to the proper balancing of the resistances of these arms, and this may require otherwise superfluous additions. It would seem that a practical solution may be attained, though it must remain deficient in mathematical exactness. The junctions of the various pieces can be effected quite satisfactorily with sealing-wax used sparingly. The brass U itself is mounted at the end of a rod held horizontally in front of the observer and parallel to the direction of motion. I found it best to work indoors in a long room or gallery.

Although in use the needle is approximately vertical, it is necessary to eliminate the possible effect of gravity more completely than can thus be attained. When the apparatus is otherwise complete, it is turned so as to make the needle horizontal, and small balance weights (finally of wax) adjusted behind the plates until equilibrium is neutral. In this process a good opinion can be formed respecting the freedom of movement.

In an experiment, suggested by the case of the mirror above referred to, the comparison was between a rectangular plate 2 inches × $1\frac{1}{2}$ inches and an elongated strip ·51 inch broad, the length of the strip being parallel to the needle, *i.e.* vertical in use. At first this length was a little in excess, but was cut down until the resistance balance was attained. For this purpose it seemed that equal areas were required to an accuracy of about one per cent., nearly on the limit set by the delicacy of the apparatus.

According to the principle of similitude the influence of linear scale (l) upon the mean pressure should enter only as a function of ν/Vl, where ν is the kinematic viscosity of air and V the velocity of travel. In the present case $\nu = {\cdot}1505$, V (4 miles per hour) $= 180$, and l, identified with the width of the strip, $= 1{\cdot}27$, all in C.G.S. measure. Thus

$$\nu/Vl = {\cdot}00066.$$

In view of the smallness of this quantity, it is not surprising that the influence of linear scale should fail to manifest itself.

In virtue of the more complete symmetry realizable when the plates to be compared are not merely equal in area but also similar in shape, this method would be specially advantageous for the investigation of the possible influence of thickness and of the smoothness of the surfaces.

When the areas to be compared are unequal, so that their centres need to be at different distances from the axis, the resistance balance of the auxiliary parts demands special attention. I have experimented upon circular disks whose areas are as 2 : 1. When there was but one smaller disk (6 cm. in diameter) the arms of the lever had to be also as 2 : 1 (fig. 1). In another

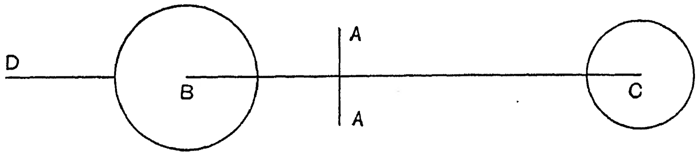

Fig. 1.

experiment *two* small disks (each 4 cm. in diameter) were balanced against a larger one of equal total area (fig. 2). Probably this arrangement is the better. In neither case was any difference of mean pressures detected.

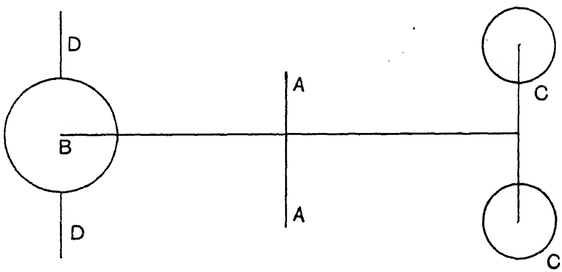

Fig. 2.

In the figures *AA* represents the needle, *B* and *C* the large and small disks respectively, *D* the extra attachments needed for the resistance balance of the auxiliary parts.

HYDRODYNAMICAL PROBLEMS SUGGESTED BY PITOT'S TUBES.

[*Proceedings of the Royal Society*, A, Vol. XCI. pp. 503—511, 1915.]

THE general use of Pitot's tubes for measuring the velocity of streams suggests hydrodynamical problems. It can hardly be said that these are of practical importance, since the action to be observed depends simply upon Bernoulli's law. In the interior of a long tube of any section, closed at the further end and facing the stream, the pressure must be that due to the velocity (v) of the stream, *i.e.* $\frac{1}{2}\rho v^2$, ρ being the density. At least, this must be the case if viscosity can be neglected. I am not aware that the influence of viscosity here has been detected, and it does not seem likely that it can be sensible under ordinary conditions. It would enter in the combination ν/vl, where ν is the kinematic viscosity and l represents the linear dimension of the tube. Experiments directed to show it would therefore be made with small tubes and low velocities.

In practice a tube of circular section is employed. But, even when viscosity is ignored, the problem of determining the motion in the neighbourhood of a circular tube is beyond our powers. In what follows, not only is the fluid supposed frictionless, but the circular tube is replaced by its two-dimensional analogue, *i.e.* the channel between parallel plane walls. Under this head two problems naturally present themselves.

The first problem proposed for consideration may be defined to be the flow of electricity in two dimensions, when the uniformity is disturbed by the presence of a channel whose infinitely thin non-conducting walls are parallel to the flow. By themselves these walls, whether finite or infinite, would cause no disturbance; but the channel, though open at the finite end, is supposed to be closed at an infinite distance away, so that, on the whole, there is no stream through it. If we suppose the flow to be of liquid instead of electricity, the arrangement may be regarded as an idealized Pitot's tube,

although we know that, in consequence of the sharp edges, the electrical law would be widely departed from. In the recesses of the tube there is no motion, and the pressure developed is simply that due to the velocity of the stream.

The problem itself may be treated as a modification of that of Helmholtz*, where flow is imagined to take place within the channel and to come to evanescence outside at a distance from the mouth. If in the usual notation† $z=x+iy$, and $w=\phi+i\psi$ be the complex potential, the solution of Helmholtz's problem is expressed by

$$z=w+e^w, \qquad \text{.......................(1)}$$

or

$$x=\phi+e^\phi \cos\psi, \qquad y=\psi+e^\phi \sin\psi. \qquad \text{...............(2)}$$

The walls correspond to $\psi=\pm\pi$, where y takes the same values, and they extend from $x=-\infty$ to $x=-1$. Also the stream-line $\psi=0$ makes $y=0$, which is a line of symmetry. In the recesses of the channel ϕ is negative and large, and the motion becomes a uniform stream.

To annul the internal stream we must superpose upon this motion, expressed say by $\phi_1+i\psi_1$, another of the form $\phi_2+i\psi_2$ where

$$\phi_2+i\psi_2=-x-iy.$$

In the resultant motion,

$$\phi=\phi_1+\phi_2=\phi_1-x, \qquad \psi=\psi_1+\psi_2=\psi_1-y,$$

so that

$$\phi_1=\phi+x, \qquad \psi_1=\psi+y,$$

and we get

$$0=\phi+e^{\phi+x}\cos(\psi+y), \qquad 0=\psi+e^{\phi+x}\sin(\psi+y), \quad \text{......} \quad (3)$$

whence

$$x=-\phi+\log\sqrt{(\phi^2+\psi^2)}, \qquad y=-\psi+\tan^{-1}(\psi/\phi) \quad \text{.........(4)}$$

or, as it may also be written,

$$z=-w+\log w. \qquad \text{.............................(5)}$$

It is easy to verify that these expressions, no matter how arrived at, satisfy the necessary conditions. Since x is an even function of ψ, and y an odd function, the line $y=0$ is an axis of symmetry. When $\psi=0$, we see from (3) that $\sin y=0$, so that $y=0$ or $\pm\pi$, and that $\cos y$ and ϕ have opposite signs. Thus when ϕ is negative, $y=0$; and when ϕ is positive, $y=\pm\pi$. Again, when ϕ is negative, x ranges from $+\infty$ to $-\infty$; and when ϕ is positive x ranges from $-\infty$ to -1, the extreme value at the limit of the wall, as appears from the equation

$$dx/d\phi=-1+1/\phi=0,$$

making $\phi=1$, $x=-1$. The central stream-line may thus be considered to pass along $y=0$ from $x=\infty$ to $x=-\infty$. At $x=-\infty$ it divides into two

* *Berlin Monatsber.* 1868; *Phil. Mag.* Vol. XXXVI. p. 337 (1868). In this paper a new path was opened.

† See Lamb's *Hydrodynamics*, § 66.

branches along $y = \pm\pi$. From $x = -\infty$ to $x = -1$, the flow is along the inner side of the walls, and from $x = -1$ to $x = -\infty$ back again along the outer side. At the turn the velocity is of course infinite.

We see from (4) that when ψ is given the difference in the final values of y, corresponding to infinite positive and negative values of ϕ, amounts to π, and that the smaller is ψ the more rapid is the change in y.

The corresponding values of x and y for various values of ϕ, and for the stream-lines $\psi = -1, -\frac{1}{2}, -\frac{1}{4}$, are given in Table I, and the more important parts are exhibited in the accompanying plots (fig. 1).

TABLE I.

ϕ	$\psi = -\frac{1}{4}$		$\psi = -\frac{1}{2}$		$\psi = -1$	
	x	y	x	y	x	y
−10	12·303	0·2750	12·30	0·550	12·31	1·100
− 5	6·610	0·3000	6·614	0·600	6·63	1·198
− 3	4·102	0·3333	4·112	0·665	4·15	1·322
− 2	2·701	0·3745	2·723	0·745	2·80	1·464
− 1	1·030	0·495	1·111	0·964	1·35	1·785
− 0·50	0·081	0·714	0·153	1·285	—	—
− 0·25	−0·790	1·035	—	—	—	—
0·00	−1·386	1·821	−0·693	2·071	0·00	2·571
0·25	−1·290	2·606	—	—	—	—
0·50	−1·081	2·928	−0·847	2·881	−0·388	3·035
1·0	−0·970	3·147	−0·888	3·178	−0·653	3·356
2·0	−1·299	3·267	−1·277	3·397	−1·195	3·678
3·0	−1·898	3·308	−1·888	3·477	—	—
4·0	—	—	—	—	−2·584	3·897
5·0	−3·389	3·342	−3·386	3·542	—	—
10·0	−7·697	3·367	—	—	−7·692	4·042
20·0	—	—	—	—	−17·00	4·092

In the second form of the problem we suppose, after Helmholtz and Kirchhoff, that the infinite velocity at the edge, encountered when the fluid adheres to the wall, is obviated by the formation of a surface of discontinuity where the condition to be satisfied is that of constant pressure and velocity. It is, in fact, a particular case of one treated many years ago by Prof. Love, entitled "Liquid flowing against a disc with an elevated rim," when the height of the rim is made infinite*. I am indebted to Prof. Love for the form into which the solution then degrades. The origin O' (fig. 2) of $x + iy$ or z is taken at one edge. The central stream-line ($\psi = 0$) follows the line of symmetry AB from $y = +\infty$ to $y = -\infty$. At $y = -\infty$ it divides, one half following the inner side of the wall CO' from $y = -\infty$ to $y = 0$, then becomes a free surface $O'D$ from $y = 0$ to $y = -\infty$. The connexion between

* *Camb. Phil. Proc.* Vol. VII. p. 185 (1891).

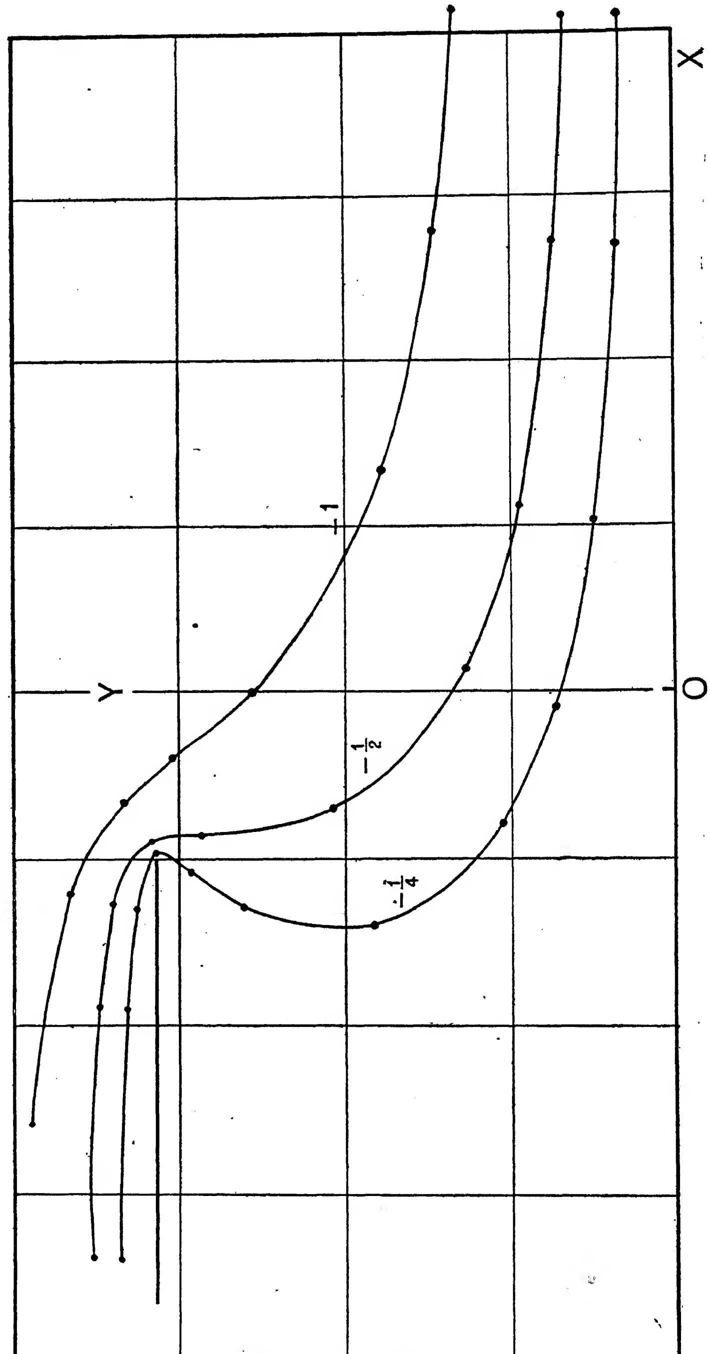

Fig. 1.

z and $w (= \phi + i\psi)$ is expressed with the aid of an auxiliary variable θ. Thus

$$z = \tan\theta - \theta - \tfrac{1}{4}i\tan^2\theta - i\log\cos\theta, \quad \text{.................(6)}$$

$$w = \tfrac{1}{4}\sec^2\theta. \quad \text{.................................(7)}$$

If we put $\tan\theta = \xi + i\eta$, we get

$$w = \tfrac{1}{4}(1 + \xi^2 - \eta^2 + 2i\xi\eta),$$

so that
$$\phi = \tfrac{1}{4}(1 + \xi^2 - \eta^2), \qquad \psi = \tfrac{1}{2}\xi\eta. \quad \text{....................(8)}$$

We find further (Love),

$$z = \xi + i\eta + \tfrac{1}{2}\xi\eta - \frac{i}{4}(\xi^2 - \eta^2) - \tfrac{1}{2}\tan^{-1}\frac{2\xi}{1 - \xi^2 - \eta^2} - \tfrac{1}{2}\tan^{-1}\frac{2\xi\eta}{1 + \xi^2 - \eta^2}$$
$$+ \frac{i}{2}\log\{(1 - \eta)^2 + \xi^2\} \quad \text{........(9)}$$

so that
$$x = \xi + \psi + \tfrac{1}{2}\tan^{-1}\frac{2\xi}{\eta^2 + \xi^2 - 1} + \tfrac{1}{2}\tan^{-1}\frac{4\psi}{\eta^2 - \xi^2 - 1}, \quad \text{......(10)}$$

$$y = \eta - \tfrac{1}{4}(\xi^2 - \eta^2) + \tfrac{1}{2}\log\{(1 - \eta)^2 + \xi^2\}. \quad \text{....................(11)}$$

The stream-lines, corresponding to a constant ψ, may be plotted from (10), (11), if we substitute $2\psi/\xi$ for η and regard ξ as the variable parameter. Since by (8)

$$\phi = \tfrac{1}{4}(1 + \xi^2) - \psi^2/\xi^2, \qquad d\phi/d\xi = \tfrac{1}{2}\xi + 2\psi^2/\xi^3,$$

there is no occasion to consider negative values of ξ, and ϕ and ξ vary always in the same direction.

As regards the fractions under the sign of $\tan^{-1}$, we see that both vanish when $\xi = 0$, and also when $\xi = \infty$. The former, viz., $2\xi \div (4\psi^2/\xi^2 + \xi^2 - 1)$, at first + when ξ is very small, rises to ∞ when $\xi^2 = \tfrac{1}{2}\{1 \pm \sqrt{(1 - 16\psi^2)}\}$, which happens when $\psi < \tfrac{1}{4}$, but not otherwise. In the latter case the fraction is always positive. When $\psi < \tfrac{1}{4}$, the fraction passes through ∞, there changing sign. The numerically least negative value is reached when $\xi^2 = \tfrac{1}{2}\{\sqrt{(1 + 48\psi^2)} - 1\}$. The fraction then retraces its entire course, until it becomes zero again when $\xi = \infty$. On the other hand the second fraction, at first positive, rises to infinity in all cases when $\xi^2 = \tfrac{1}{2}\{\sqrt{(1 + 16\psi^2)} - 1\}$, after which it becomes negative and decreases numerically to zero, no part of its course being retraced. As regards the ambiguities in the resulting angles, it will suffice to suppose both angles to start from zero with ξ. This choice amounts to taking the origin of x at O, instead of O'.

When ψ is very small the march of the functions is peculiar. The first fraction becomes infinite when $\xi^2 = 4\psi^2$, that is when ξ is still small. The turn occurs when $\xi^2 = 12\psi^2$, and the corresponding least negative value is also small. The first $\tan^{-1}$ thus passes from 0 to π while ξ is still small. The second fraction also becomes infinite when $\xi^2 = 4\psi^2$, there changing sign, and again approaches zero while ξ is of the same order of magnitude.

The second $\tan^{-1}$ thus passes from 0 to π, thereby completing its course, while ξ is still small.

When $\psi = 0$ absolutely, either ξ or η, or both, must vanish, but we must still have regard to the relative values of ψ and ξ. Thus when ξ is small *enough*, $x = 0$, and this part of the stream-line coincides with the axis of symmetry. But while ξ is still small, x changes from 0 to π, the new value representing the inner face of the wall. The transition occurs when $\xi = 2\psi$, $\eta = 1$, making in (11) $y = -\infty$. The point O' at the edge of the wall ($x = \pi$, $y = 0$) corresponds to $\xi = 0$, $\eta = 0$.

For the free part of the stream-line we may put $\eta = 0$, so that

$$x = \xi + \tfrac{1}{2}\tan^{-1}\frac{2\xi}{\xi^2 - 1} + \frac{\pi}{2} = \xi - \tan^{-1}\xi + \pi, \quad \text{............(12)}$$

where $\tan^{-1}\xi$ is to be taken between 0 and $\frac{1}{2}\pi$. Also

$$y = -\tfrac{1}{4}\xi^2 + \tfrac{1}{2}\log(1 + \xi^2). \quad \text{........................(13)}$$

When ξ is very great,

$$x = \xi + \tfrac{1}{2}\pi, \qquad y = -\tfrac{1}{4}\xi^2, \quad \text{........................(14)}$$

and the curve approximates to a parabola.

When ξ is small,

$$x - \pi = \tfrac{1}{3}\xi^3, \qquad y = \tfrac{1}{4}\xi^2, \quad \text{........................(15)}$$

so that the ratio $(x - \pi)/y$ starts from zero, as was to be expected.

The upward movement of y is of but short duration. It may be observed that, while $dx/d\xi$ is always positive,

$$\frac{dy}{d\xi} = \frac{\xi(1 - \xi^2)}{2(1 + \xi^2)}, \quad \text{............................(16)}$$

which is positive only so long as $\xi < 1$. And when $\xi = 1$,

$$x - \pi = 1 - \tfrac{1}{4}\pi = 0{\cdot}2146, \qquad y = -\tfrac{1}{4} + \log 2 = 0{\cdot}097.$$

Some values of x and y calculated from (12), (13) are given in Table II and the corresponding curve is shown in fig. 3.

TABLE II.—$\psi = 0$.

ξ	x	y	ξ	x	y
0·0	3·142	0	2·5	4·451	− 0·571
0·5	3·178	+0·050	3·0	4·892	− 1·098
1·0	3·356	+0·097	4·0	5·816	− 2·583
1·5	3·659	+0·027	5·0	6·768	− 4·62
2·0	4·034	−0·195	20·0	21·621	−97·00

It is easy to verify that the velocity is constant along the curve defined by (12), (13). We have

$$\frac{dx}{d\phi} = \frac{\xi^2}{1+\xi^2}\frac{d\xi}{d\phi}, \qquad \frac{dy}{d\phi} = \frac{\xi}{2}\frac{1-\xi^2}{1+\xi^2}\frac{d\xi}{d\phi};$$

and when $\psi = 0$,

$$\phi = \tfrac{1}{4}(1+\xi^2), \qquad d\phi/d\xi = \tfrac{1}{2}\xi.$$

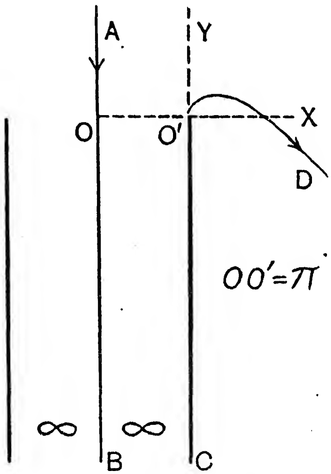

Fig. 2.

Thus
$$\frac{dx}{d\phi} = \frac{2\xi}{1+\xi^2}, \qquad \frac{dy}{d\phi} = \frac{1-\xi^2}{1+\xi^2},$$

and
$$(dx/d\phi)^2 + (dy/d\phi)^2 = 1. \quad \text{.......................(17)}$$

The square root of the expression on the left of (17) represents the reciprocal of the resultant velocity.

TABLE III.—$\psi = \frac{1}{10}$.

ξ	x	y	ξ	x	y
0	0	∞	0·40	2·9667	+0·076
0·05	0·1667	9·098	0·50	3·0467	0·130
0·10	0·2995	3·008	0·60	3·1089	0·162
0·13	0·4668	1·535	0·80	3·2239	0·198
0·15	0·6725	0·766	1·00	3·3454	0·207
0·17	1·0368	+0·109	1·50	3·6947	+0·125
0·18	1·2977	−0·143	2·00	4·0936	−0·112
0·19	1·5907	−0·304	2·50	4·5234	−0·501
0·20	1·8708	−0·370	3·00	4·9725	−1·032
0·22	2·2828	−0·331	4·00	5·9039	−2·536
0·25	2·5954	−0·195	6·00	7·8305	−7·161
0·30	2·8036	−0·047			

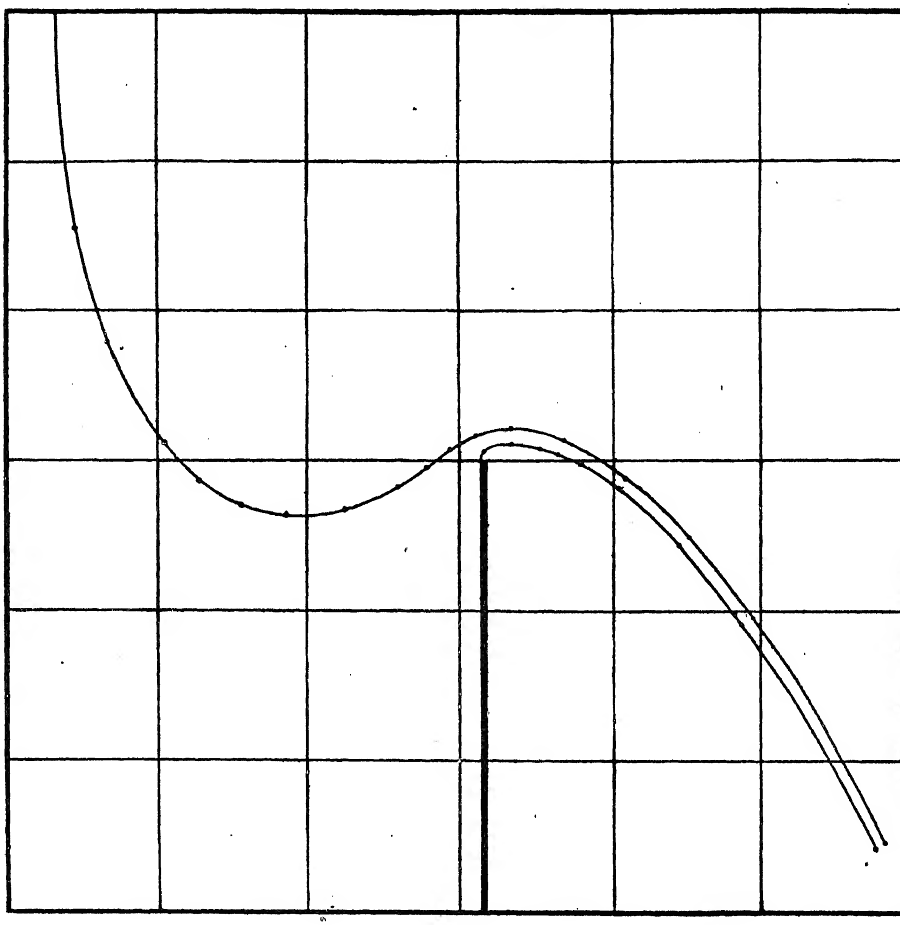

Fig. 3.

When ψ differs from zero, the calculations are naturally more complicated. The most interesting and instructive cases occur when ψ is small. I have chosen $\psi = 1/10$. The corresponding values of ξ, x, and y are given in Table III, calculated from equations (10), (11), and a plot is shown in fig. 3.

As in the former problem, where the liquid is supposed to adhere to the walls notwithstanding the sharp edges, the pressure in the recesses of the tube is simply that due to the velocity at a distance. At other places the pressure can be deduced from the stream-function in the usual way.

397.

ON THE CHARACTER OF THE "S" SOUND.

[*Nature*, Vol. XCV. pp. 645, 646, 1915.]

SOME two years ago I asked for suggestions as to the formation of an artificial hiss, and I remarked that the best I had then been able to do was by blowing through a rubber tube nipped at about half an inch from the open end with a screw clamp, but that the sound so obtained was perhaps more like an *f* than an *s*. "There is reason to think that the ear, at any rate of elderly people, tires rapidly to a maintained hiss. The pitch is of the order of 10,000 per second*." The last remark was founded upon experiments already briefly described† under the head "Pitch of Sibilants."

"Doubtless this may vary over a considerable range. In my experiments the method was that of nodes and loops (*Phil. Mag.* Vol. VII. p. 149 (1879); *Scientific Papers*, Vol. I. p. 406), executed with a sensitive flame and sliding reflector. A hiss given by Mr Enock, which to me seemed very high and not over audible, gave a wave length (λ) equal to 25 mm., with good agreement on repetition. A hiss which I gave was graver and less definite, corresponding to $\lambda = 32$ mm. The frequency would be of the order of 10,000 per second, more than 5 octaves above middle C."

Among the replies publicly or privately given, with which I was favoured, was one from Prof. E. B. Titchener, of Cornell University‡, who wrote:

"Lord Rayleigh's sound more like an *f* than an *s* is due, according to Köhler's observations, to a slightly too high pitch. A Galton whistle, set for a tone of 8400 v.d., will give a pure *s*."

It was partly in connexion with this that I remarked later§ that I doubted whether any pure tone gives the full impression of an *s*, having often experimented with bird-calls of about the right pitch. In my published papers I

* *Nature*, Vol. XCI. p. 319, 1913.

† *Phil. Mag.* Vol. XVI. p. 235, 1908; *Scientific Papers*, Vol. V. p. 486.

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find references to wave-lengths 31·2 mm., 1·304 in. = 33·1 mm., 1·28 in. = 32·5 mm.* It is true that these are of a pitch too high for Köhler's optimum, which at ordinary temperatures corresponds to a wave-length of 40·6 mm., or 1·60 inches; but they agree pretty well with the pitch found for actual hisses in my observations with Enock.

Prof. Titchener has lately returned to the subject. In a communication to the American Philosophical Society† he writes:

"It occurred to me that the question might be put to the test of experiment. The sound of a Galton's whistle set for 8400 v.d. might be imitated by the mouth, and a series of observations might be taken upon material composed partly of the natural (mouth) sounds and partly of the artificial (whistle) tones. If a listening observer were unable to distinguish between the two stimuli, and if the mouth sound were shown, phonetically, to be a true hiss, then it would be proved that the whistle also gives an *s*, and Lord Rayleigh would be answered.

"The experiment was more troublesome than I had anticipated; but I may say at once that it has been carried out, and with affirmative result."

A whistle of Edelmann's pattern (symmetrical, like a steam whistle) was used, actuated by a rubber bulb; and it appears clear that a practised operator was able to imitate the whistle so successfully that the observer could not say with any certainty which was which. More doubt may be felt as to whether the sound was really a fully developed hiss. Reliance seems to have been placed almost exclusively upon the position of the lips and tongue of the operator. I confess I should prefer the opinion of unsophisticated observers judging of the result simply by ear. The only evidence of this kind mentioned is in a footnote (p. 328): "Mr Stephens' use of the word 'hiss' was spontaneous, not due to suggestion." I have noticed that sometimes a hiss passes momentarily into what may almost be described as a whistle, but I do not think this can be regarded as a normal *s*.

Since reading Prof. Titchener's paper I have made further experiments with results that I propose to describe. The pitch of the sounds was determined by the sensitive flame and sliding reflector method, which is abundantly sensitive for the purpose. The reflector is gradually drawn back from the burner, and the positions noted in which the flame is unaffected. This phase occurs when the burner occupies a *node* of the stationary waves. It is a place where there is no to and fro *motion*. The places of recovery are thus at distances from the reflector which are (odd or even) multiples of the half wave-length. The reflector was usually drawn back until there had been five

* *Scientific Papers*, Vol. I. p. 407; Vol. II. p. 100.

† *Proceedings*, Vol. LIII. August—December, 1914, p. 323.

recoveries, indicating that the distance from the burner was now $5 \times \frac{1}{2}\lambda$, and this distance was then measured.

The first observations were upon a whistle on Edelmann's pattern of my own construction. The flame and reflector gave $\lambda = 1{\cdot}7$ in., about a semi-tone flat on Köhler's optimum. As regards the character of the sound, it seemed to me and others to bear some resemblance to an *s*, but still to be lacking in something essential. I should say that since my own hearing for *s*'s is now distinctly bad, I have always confirmed my opinion by that of other listeners whose hearing is good. That there should be some resemblance to an *s* at a pitch which is certainly the predominant pitch of an *s* is not surprising; and it is difficult to describe exactly in what the deficiency consisted. My own impression was that the sound was too nearly a pure tone, and that if it had been quite a pure tone the resemblance to an *s* would have been less. In subsequent observations the pitch was raised through $\lambda = 1{\cdot}6$ in., but without modifying the above impressions.

Wishing to try other sources which I thought more likely to give pure tones, I fell back on bird calls. A new one, with adjustable distance between the perforated plates, gave on different trials $\lambda = 1{\cdot}8$ in., $\lambda = 1{\cdot}6$ in. In neither case was the sound judged to be at all a proper *s*, though perhaps some resemblance remained. The effect was simply that of a high note, like the squeak of a bird or insect. Further trials on another day gave confirmatory results.

The next observations were made with the highest pipe from an organ, gradually raised in pitch by cutting away at the open end. There was some difficulty in getting quite high enough, but measures were taken giving $\lambda = 2{\cdot}2$ in., $\lambda = 1{\cdot}9$ in., and eventually $\lambda = 1{\cdot}6$ in. In no case was there more than the slightest suggestion of an *s*.

As I was not satisfied that at the highest pitch the organ-pipe was speaking properly, I made another from lead tube, which could be blown from an *adjustable* wind nozzle. Tuned to give $\lambda = 1{\cdot}6$ in., it sounded faint to my ear, and conveyed no *s*. Other observers, who heard it well, said it was no *s*.

In all these experiments the sounds were *maintained*, the various instruments being blown from a loaded bag, charged beforehand with a foot blower. In this respect they are not fully comparable with those of Prof. Titchener, whose whistle was actuated by squeezing a rubber bulb. However, I have also tried a glass tube, 10·4 in. long, supported at the middle and rubbed with a resined leather. This should be of the right pitch, but the squeak heard did not suggest an *s*. I ought perhaps to add that the thing did not work particularly well.

It will be seen that my conclusions differ a good deal from those of Prof. Titchener, but since these estimates depend upon individual judgment, perhaps

not uninfluenced by prepossessions, they are not fully satisfactory. Further independent aural observations are desirable. I fear a record, or ocular observation, of vibrations at so high a pitch is hardly feasible.

I may perhaps be asked if a characteristic *s*, having a dominant pitch, is not a pure tone, what is it? I am disposed to think that the vibration is irregular. A fairly defined pitch does not necessitate regular sequences of more than a few (say 3—10) vibrations. What is the state of affairs in an organ-pipe which does not speak well, or in a violin string badly bowed? An example more amenable to observation is afforded by the procession of drops into which a liquid jet breaks up. If the jet is well protected from outside influences, the procession is irregular, and yet there is a dominant interval between consecutive drops, giving rise under suitable conditions to a sound having a dominant pitch. Vibrations of this sort deserve more attention than they have received. In the case of the *s* the pitch is so high that there would be opportunity for interruptions so frequent that they would not be separately audible, and yet not so many as to preclude a fairly defined dominant pitch. I have an impression, too, that the *s* includes subordinate components decidedly graver than the dominant pitch.

Similar questions naturally arise over the character of the *sh*, *f*, and *th* sounds.

ON THE STABILITY OF THE SIMPLE SHEARING MOTION OF A VISCOUS INCOMPRESSIBLE FLUID.

[*Philosophical Magazine*, Vol. XXX. pp. 329–338, 1915.]

A PRECISE formulation of the problem for free infinitesimal disturbances was made by Orr (1907)*. It is supposed that ζ (the vorticity) and v (the velocity perpendicular to the walls) are proportional to $e^{int}e^{ikx}$, where $n = p + iq$. If $\nabla^2 v = S$, we have

$$\frac{d^2S}{dy^2} = \left\{k^2 - \frac{q}{\nu} + \frac{i}{\nu}(p + k\beta y)\right\} S, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$d^2v/dy^2 - k^2 v = S, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

with the boundary conditions that $v = 0$, $dv/dy = 0$ at the walls where y is constant. Here ν is the kinematic viscosity, and β is proportional to the initial constant vorticity. Orr easily shows that the period-equation takes the form

$$\int S_1 e^{ky}dy \,.\int S_2 e^{-ky}dy - \int S_1 e^{-ky}dy \,.\int S_2 e^{ky}dy = 0, \quad \ldots\ldots\ldots(3)$$

where S_1, S_2 are any two independent solutions of (1) and the integrations are extended over the interval between the walls. An equivalent equation was given a little later (1908) independently by Sommerfeld.

Stability requires that for no value of k shall any of the q's determined by (3) be negative. In his discussion Orr arrives at the conclusion that this condition is satisfied. Another of Orr's results may be mentioned. He shows that $p + k\beta y$ necessarily changes sign in the interval between the walls†.

In the paper quoted reference was made also to the work of v. Mises and Hopf, and it was suggested that the problem might be simplified if it could be shown that $q - \nu k^2$ cannot vanish. If so, it will follow that q is always

* *Proc. Roy. Irish Acad.* Vol. XXVII.

† *Phil. Mag.* Vol. XXVIII. p. 618 (1914).

positive and indeed greater than νk^2, inasmuch as this is certainly the case when $\beta=0$*. The assumption that $q=\nu k^2$, by which the real part of the $\{\ \}$ in (1) disappears, is indeed a considerable simplification, but my hope that it would lead to an easy solution of the stability problem has been disappointed. Nevertheless, a certain amount of progress has been made which it may be desirable to record, especially as the preliminary results may have other applications.

If we take a real η such that

$$p+k\beta y=-(9\nu k^2\beta^2)^{\frac{1}{3}}\,\eta, \quad\ldots\ldots\ldots\ldots(4)$$

we obtain

$$\frac{d^2S}{d\eta^2}=-9i\eta S. \quad\ldots\ldots\ldots\ldots(5)$$

This is the equation discussed by Stokes in several papers†, if we take x in his equation (18) to be the pure imaginary $i\eta$.

The boundary equation (3) retains the same form with $e^{\lambda\eta}\,d\eta$ for $e^{ky}\,dy$, where

$$\lambda^3=9\nu k^2/\beta. \quad\ldots\ldots\ldots\ldots(6)$$

In (5), (6) η and λ are non-dimensional.

Stokes exhibits the general solution of the equation

$$\frac{d^2S}{dx^2}-9xS=0 \quad\ldots\ldots\ldots\ldots(7)$$

in two forms. In ascending series which are always convergent,

$$S=A\left\{1+\frac{9x^3}{2.3}+\frac{9^2x^6}{2.3.5.6}+\frac{9^3x^9}{2.3.5.6.8.9}+\ldots\right\}$$
$$+B\left\{x+\frac{9x^4}{3.4}+\frac{9^2x^7}{3.4.6.7}+\frac{9^3x^{10}}{3.4.6.7.9.10}+\ldots\right\}\ldots\ldots\ldots(8)$$

The alternative semi-convergent form, suitable for calculation when x is large, is

$$S=Cx^{-\frac{1}{4}}e^{-2x^{\frac{3}{2}}}\left\{1-\frac{1.5}{1.144x^{\frac{3}{2}}}+\frac{1.5.7.11}{1.2.144^2x^3}-\frac{1.5.7.11.13.17}{1.2.3.144^3x^{\frac{9}{2}}}+\ldots\right\}$$

$$+Dx^{-\frac{1}{4}}e^{2x^{\frac{3}{2}}}\left\{1+\frac{1.5}{1.144x^{\frac{3}{2}}}+\frac{1.5.7.11}{1.2.144^2x^3}+\frac{1.5.7.11.13.17}{1.2.3.144^3x^{\frac{9}{2}}}+\ldots\right\},\ldots(9)$$

in which, however, the constants C and D are liable to a discontinuity. When x is real—the case in which Stokes was mainly interested—or a pure imaginary, the calculations are of course simplified.

* *Phil. Mag.* Vol. XXXIV. p. 69 (1892); *Scientific Papers*, Vol. III. p. 583.

† Especially *Camb. Phil. Trans.* Vol. X. p. 106 (1857); *Collected Papers*, Vol. IV. p. 77.

If we take as S_1 and S_2 the two series in (8), the real and imaginary parts of each are readily separated. Thus if

$$S_1 = s_1 + it_1, \qquad S_2 = s_2 + it_2, \quad \dots\dots\dots\dots(10)$$

we have on introduction of $i\eta$

$$s_1 = 1 - \frac{9^2\eta^6}{2.3.5.6} + \frac{9^4\eta^{12}}{2.3.5.6.8.9.11.12} - \dots \quad \dots\dots(11)$$

$$t_1 = -\frac{9\eta^3}{2.3} + \frac{9^3\eta^9}{2.3.5.6.8.9} - \dots \quad \dots\dots\dots\dots(12)$$

$$s_2 = \frac{9\eta^4}{3.4} - \frac{9^3\eta^{10}}{3.4.6.7.9.10} + \dots \quad \dots\dots\dots\dots(13)$$

$$t_2 = \eta - \frac{9^2\eta^7}{3.4.6.7} + \frac{9^4\eta^{13}}{3.4.6.7.9.10.12.13} - \dots \quad \dots\dots(14)$$

in which it will be seen that s_1, s_2 are even in η, while t_1, t_2 are odd.

When $\eta < 2$, these ascending series are suitable. When $\eta > 2$, it is better to use the descending series, but for this purpose it is necessary to know the connexion between the constants A, B and C, D. For $x = i\eta$ these are (Stokes)

$$A = \pi^{-\frac12}\Gamma(\tfrac13)\{C + De^{-i\pi/6}\}, \qquad B = 3\pi^{-\frac12}\Gamma(\tfrac23)\{-C + De^{i\pi/6}\} \dots(15)$$

Thus for the first series S_1 ($A = 1$, $B = 0$ in (8))

$$\log D = 1{\cdot}5820516, \qquad C = De^{\pi i/6}; \quad \dots\dots\dots\dots(16)$$

and for S_2 ($A = 0$, $B = 1$)

$$\log D' = 1{\cdot}4012366, \qquad -C' = D'e^{-i\pi/6}, \quad \dots\dots\dots\dots(17)$$

so that if the two functions in (9) be called Σ_1 and Σ_2,

$$S_1 = C\Sigma_1 + D\Sigma_2, \qquad S_2 = C'\Sigma_1 + D'\Sigma_2. \quad \dots\dots\dots\dots(18)$$

These values may be confirmed by a comparison of results calculated first from the ascending series and secondly from the descending series when $\eta = 2$. Much of the necessary arithmetic has been given already by Stokes*. Thus from the ascending series

$$s_1(2) = -13{\cdot}33010, \qquad t_1(2) = 11{\cdot}62838;$$
$$s_2(2) = -\ \ 2{\cdot}25237, \qquad t_2(2) = -11{\cdot}44664.$$

In calculating from the descending series the more important part is Σ_1, since

$$e^{-2x^{\frac32}} = e^{-2i^{\frac32}\eta^{\frac32}} = e^{\sqrt2 . \eta^{\frac32}(1-i)}.$$

For $\eta = 2$ Stokes finds

$$\Sigma_1 = -14{\cdot}98520 + 43{\cdot}81046\,i,$$

of which the log. modulus is $1{\cdot}6656036$, and the phase $+108^\circ\ 52'\ 58''{\cdot}99$. When the multiplier C or C' is introduced, there will be an addition of $\pm 30^\circ$ to this phase. Towards the value of S_1 I find

$$-13{\cdot}32487 + 11{\cdot}63096\,i;$$

* *Loc. cit.* Appendix. It was to take advantage of this that the "9" was introduced in (5).

and towards that of S_2

$$-2{\cdot}24892 - 11{\cdot}44495\,i.$$

For the other part involving D or D' we get in like manner

$$-{\cdot}00523 - {\cdot}00258\,i,$$

and
$$-{\cdot}00345 - {\cdot}00170\,i.$$

TABLE I.

η	s_1	t_1	s_2	t_2
0·0	+ 1·0000	− ·0000	+ ·0000	+ ·0000
0·1	+ 1·0000	− ·0015	+ ·0001	+ ·1000
0·2	+ 1·0000	− ·0120	+ ·0012	+ ·2000
0·3	+ ·9997	− ·0405	+ ·0061	+ ·3000
0·4	+ ·9982	− ·0960	+ ·0192	+ ·3997
0·5	+ ·9930	− ·1874	+ ·0469	+ ·4987
0·6	+ ·9790	− ·3234	+ ·0971	+ ·5955
0·7	+ ·9393	− ·5485	+ ·1969	+ ·6845
0·8	+ ·8825	− ·7605	+ ·3055	+ ·7663
0·9	+ ·7619	− 1·0717	+ ·4865	+ ·8234
1·0	+ ·554	− 1·444	+ ·734	+ ·840
1·1	+ ·215	− 2·007	+ 1·057	+ ·790
1·2	− ·310	− 2·304	+ 1·456	+ ·634
1·3	− 1·083	− 2·707	+ 1·923	+ ·320
1·4	− 2·173	− 2·979	+ 2·424	− ·221
1·5	− 3·635	− 2·972	+ 2·893	− 1·067
1·6	− 5·493	− 2·466	+ 3·212	− 2·303
1·7	− 7·694	− 1·161	+ 3·191	− 3·998
1·8	−10·057	+ 1·325	+ 2·550	− 6·173
1·9	−12·177	+ 5·441	+ ·899	− 8·745
2·0	−13·330	+11·628	− 2·252	−11·447
2·1	−12·34	+20·19	− 7·46	−13·70
2·2	− 7·49	+31·01	−15·24	−14·50
2·3	+ 3·54	+43·20	−25·84	−12·22
2·4	+23·55	+54·54	−38·90	− 4·53
2·5	+55·20	+60·44	−52·70	+11·59

It appears that with the values of C, D, C', D' defined by (16), (17) the calculations from the ascending and descending series lead to the same results when $\eta = 2$. What is more, and it is for this reason principally that I have detailed the numbers, the second part involving Σ_2 loses its importance when η exceeds 2. Beyond this point the numbers given in the table are calculated from Σ_1 only. Thus ($\eta > 2$)

$$s_1 + it_1 = D\eta^{-\frac{1}{4}}\, e^{\sqrt{2}\,.\,\eta^{\frac{3}{2}}}\, e^{-i(\sqrt{2}\,.\,\eta^{\frac{3}{2}} + \pi/8 - \pi/6)}$$

$$\times\left\{1 - \frac{1\,.\,5}{1\,.\,144\,(i\eta)^{\frac{3}{2}}} + \frac{1\,.\,5\,.\,7\,.\,11}{1\,.\,2\,.\,144^2\,(i\eta)^3} - \ldots\right\}, \quad\ldots\ldots\ldots\ldots(19)$$

$$s_2 + it_2 = -D'\eta^{-\frac{1}{4}}\, e^{\sqrt{2}\,.\,\eta^{\frac{3}{2}}}\, e^{-i(\sqrt{2}\,.\,\eta^{\frac{3}{2}} + \pi/8 + \pi/6)}$$

$$\times\left\{1 - \frac{1\,.\,5}{1\,.\,144\,(i\eta)^{\frac{3}{2}}} + \frac{1\,.\,5\,.\,7\,.\,11}{1\,.\,2\,.\,144^2\,(i\eta)^3} - \ldots\right\}, \quad\ldots\ldots\ldots\ldots(20)$$